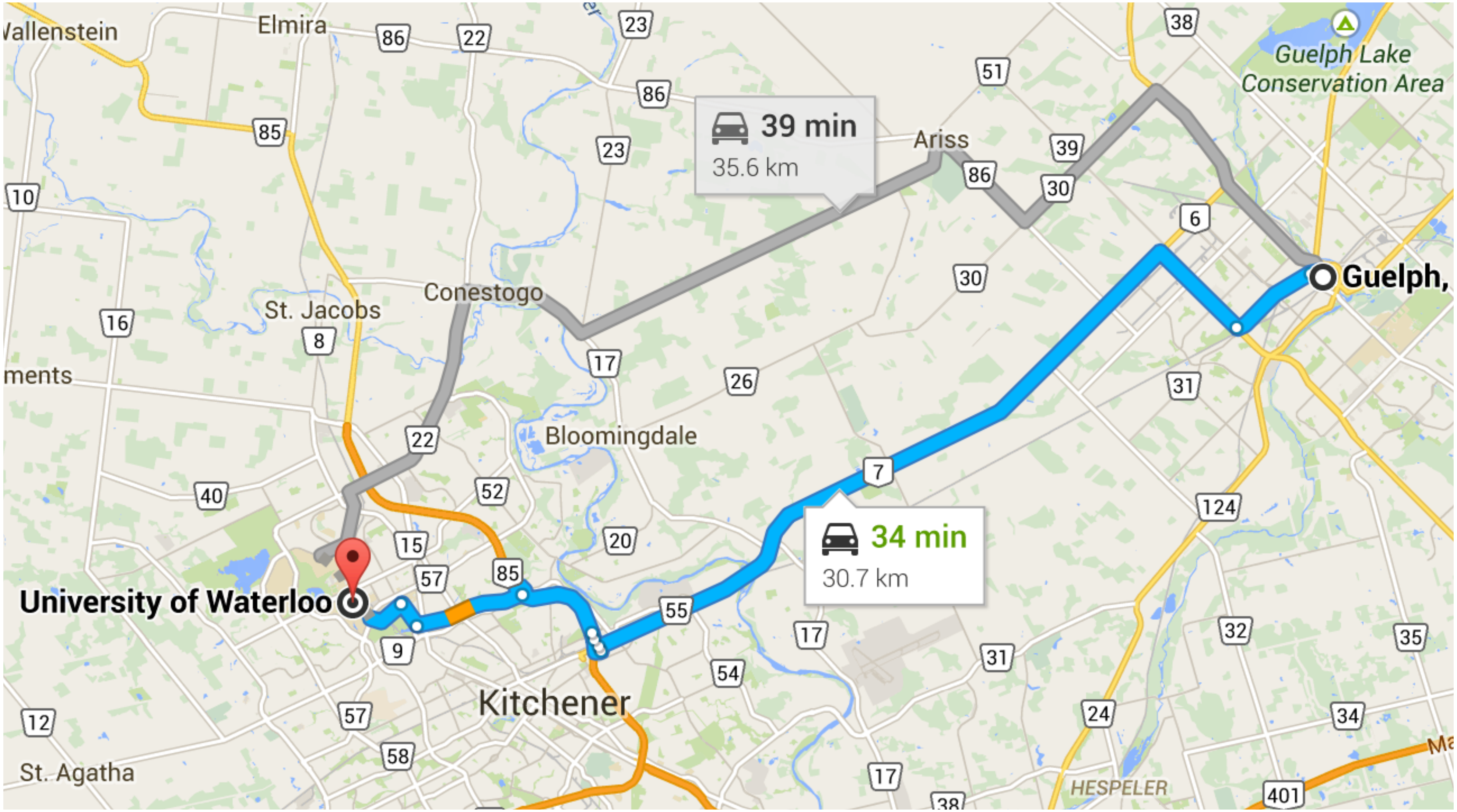
# Finding Shortest Paths

Anna Lubiw

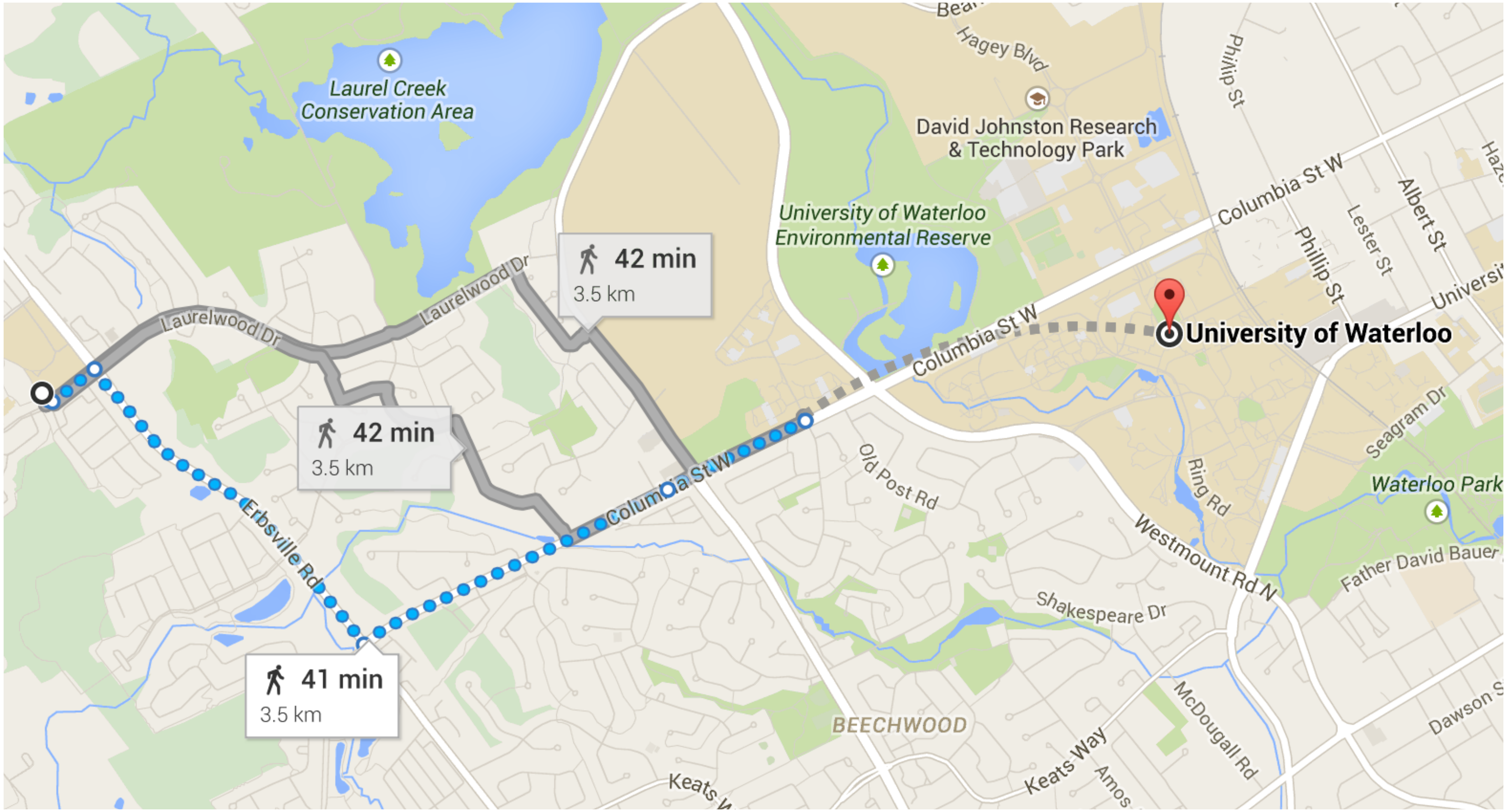
University of Waterloo

# Google Maps

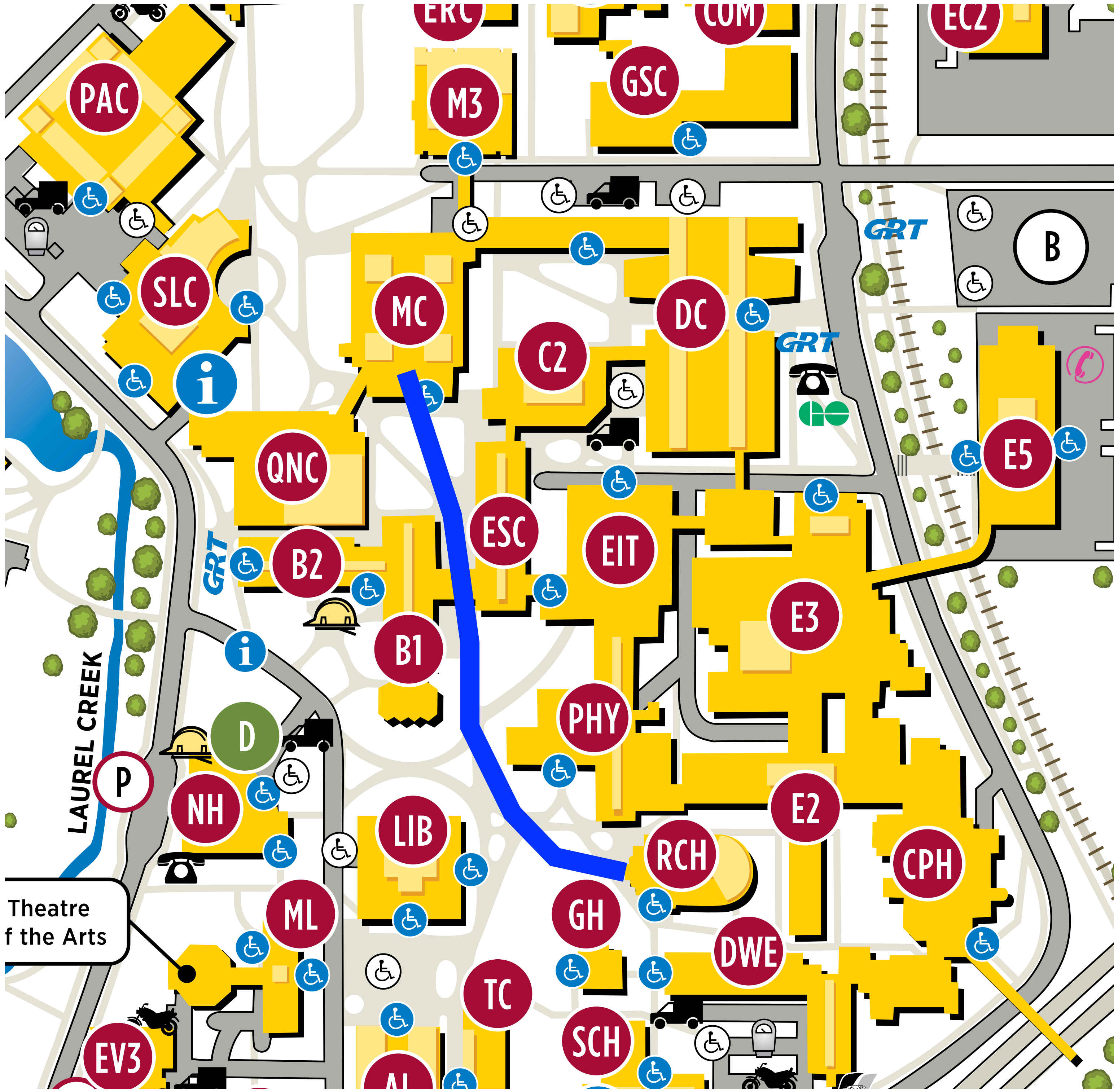


39 min
35.6 km
34 min
30.7 km
University of Waterloo
Guelph,
Kitchener
Elmira
Conestogo
St. Jacobs
Bloomingdale
Ariss
St. Agatha
HESPELER
Guelph Lake
Conservation Area

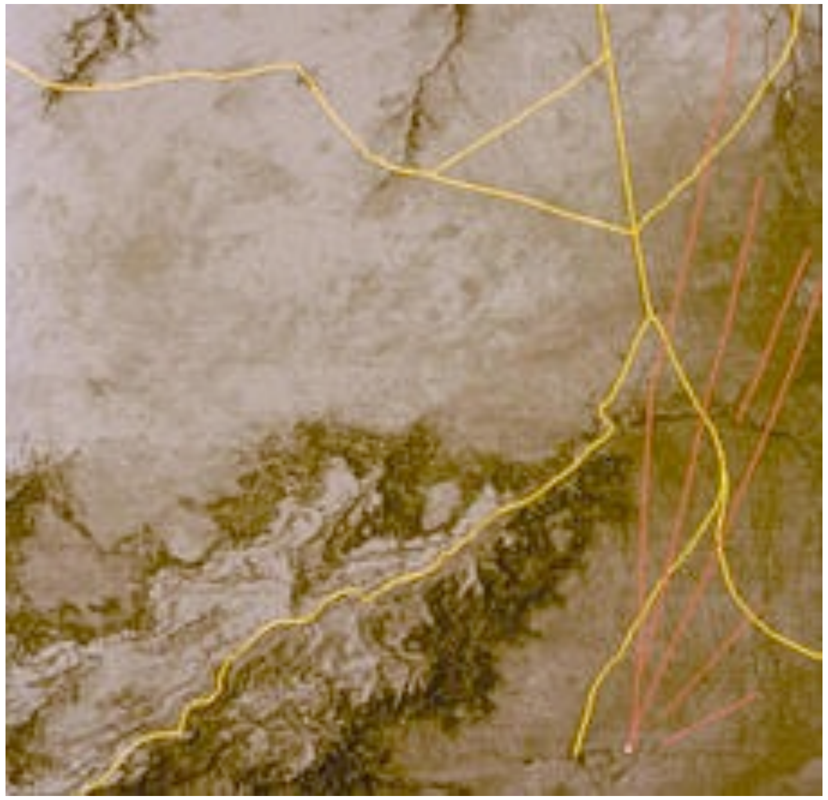
# Google Maps

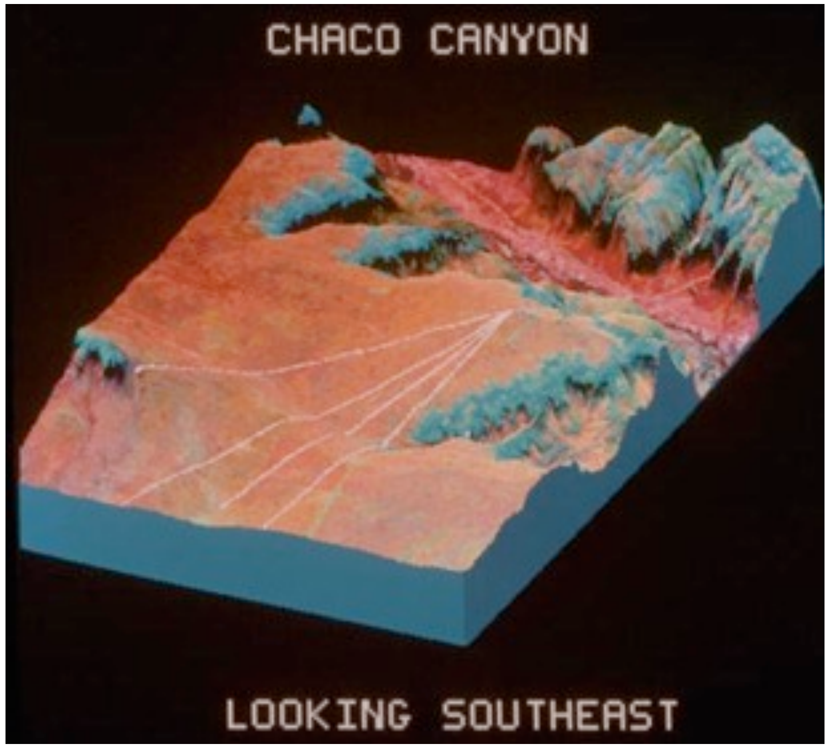

Laurel Creek Conservation Area
David Johnston Research & Technology Park
University of Waterloo Environmental Reserve
42 min
3.5 km
42 min
3.5 km
41 min
3.5 km
University of Waterloo
Laurelwood Dr
Erbsville Rd
Columbia St W
Hagey Blvd
Phillip St
Lester St
Albert St
Old Post Rd
Ring Rd
Westmount Rd N
Seagram Dr
Waterloo Park
Father David Bauer
Shakespeare Dr
BEECHWOOD
Keats Way
McDougall Rd
Dawson S

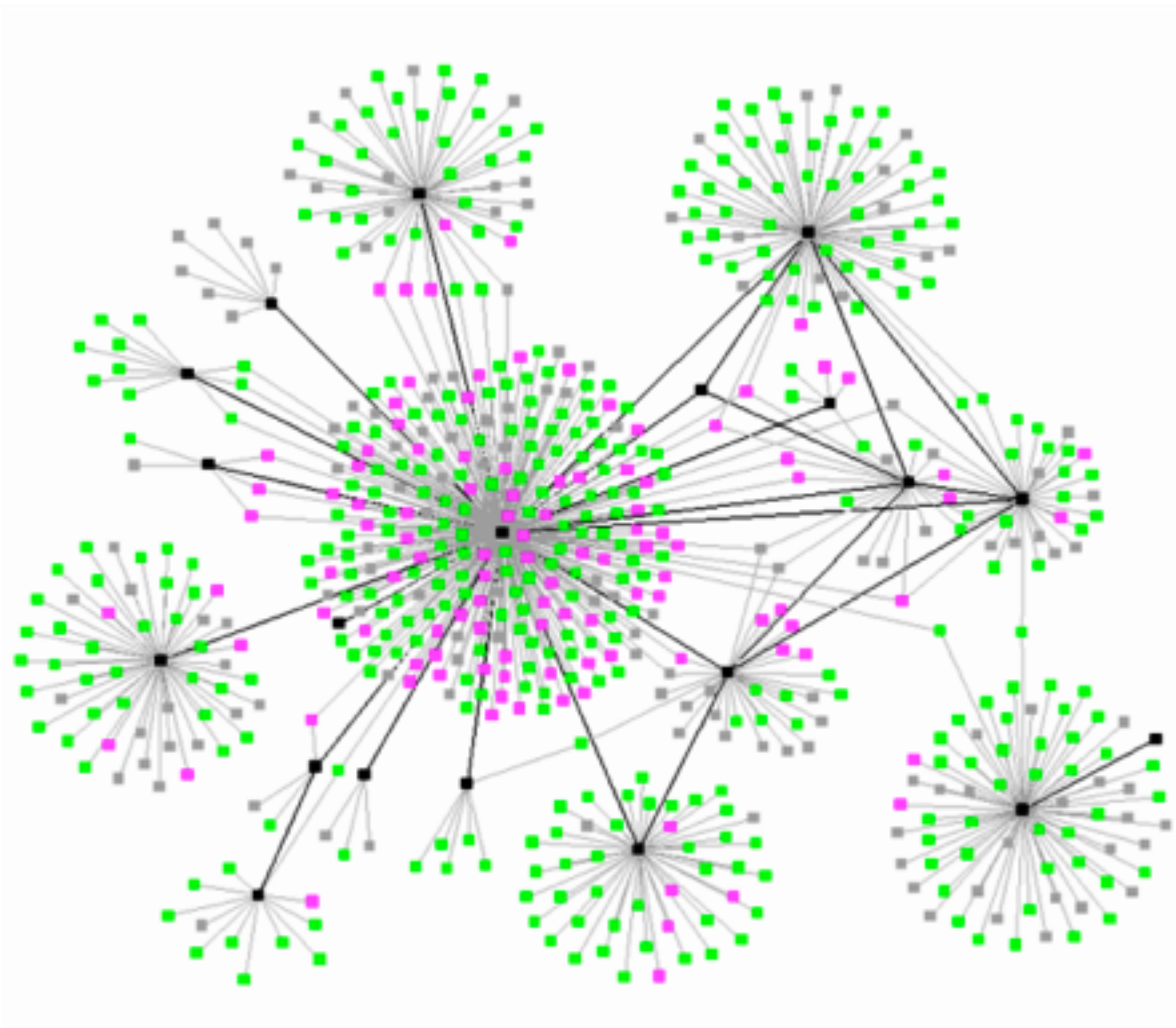
# Paths through Space



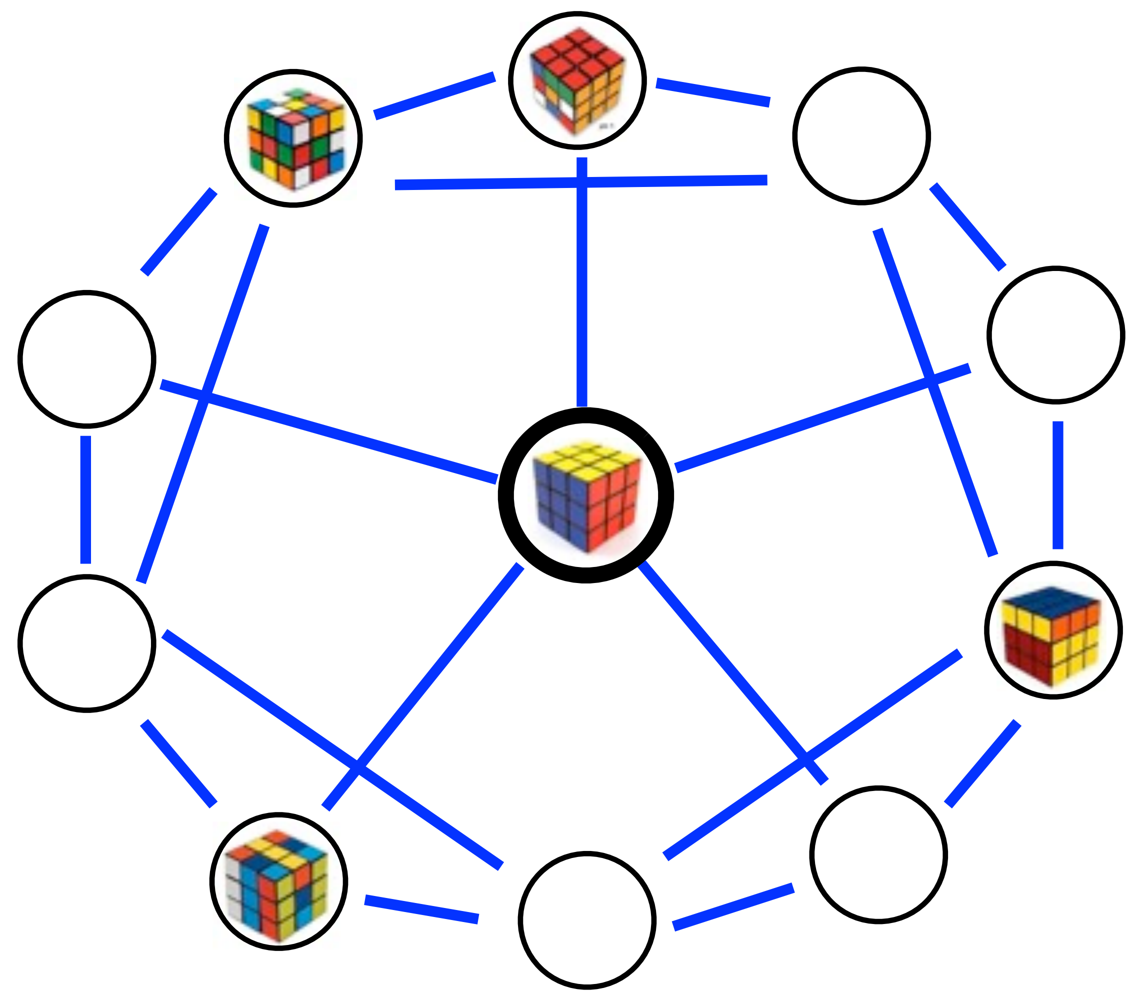
# Paths through Space





social network
6 degrees of separation



Rubik's cube
diameter = 20

# 6 Degrees of Separation (Small World)

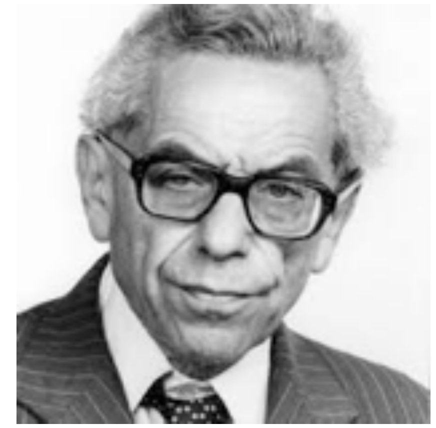
Queen Elizabeth

David Johnston


UW prof

you

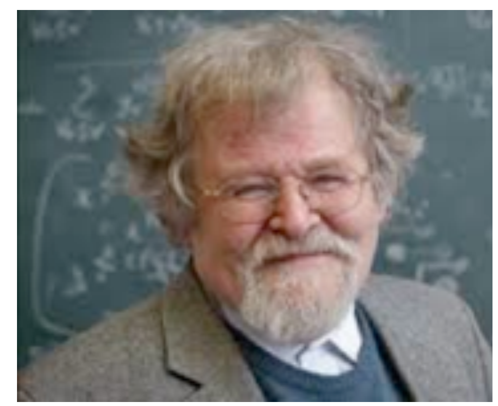
# Erdős Number



Paul Erdős

Erdos wrote papers with these 3
(and 508 others)

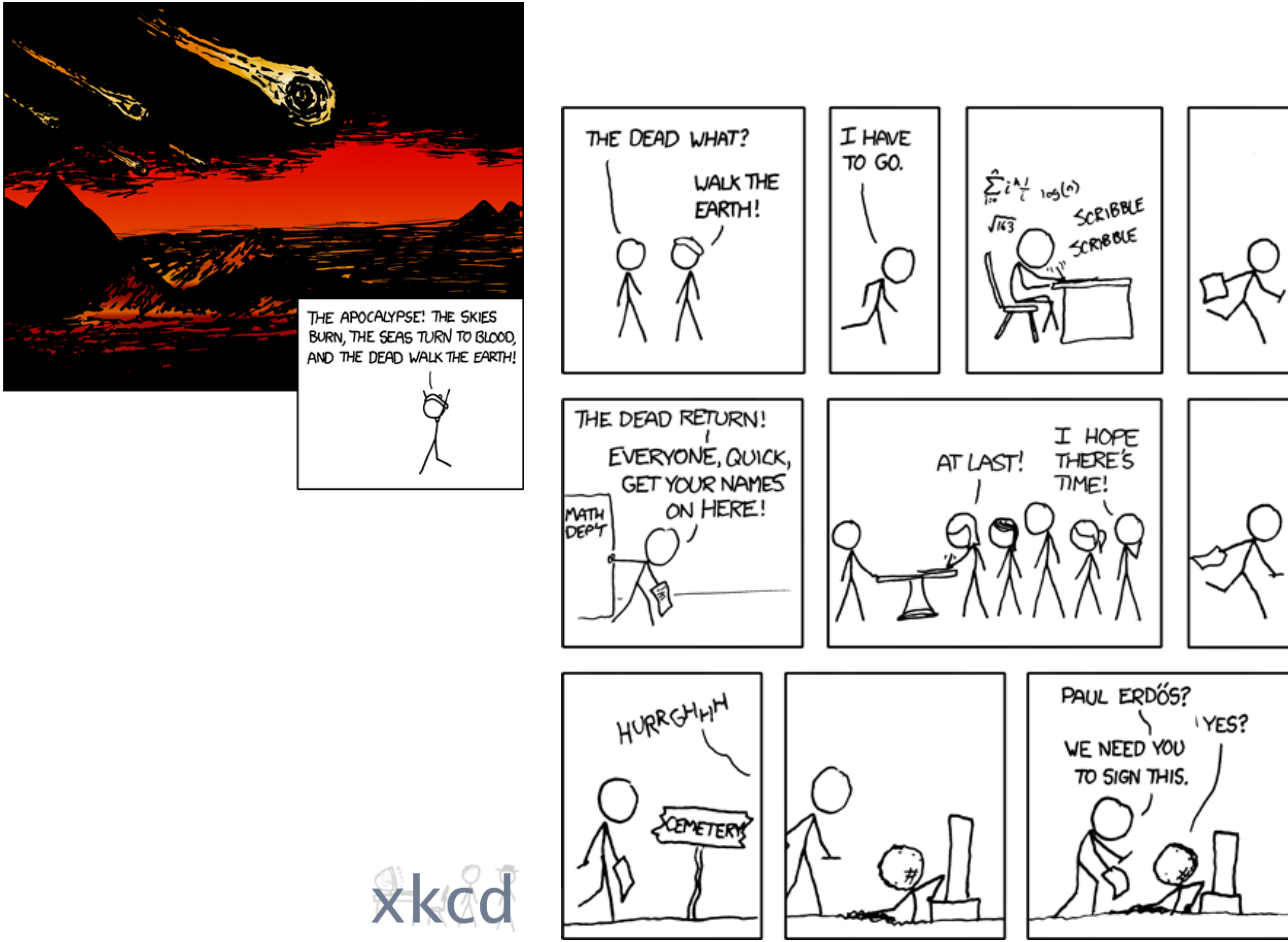
Jeffrey Shallit

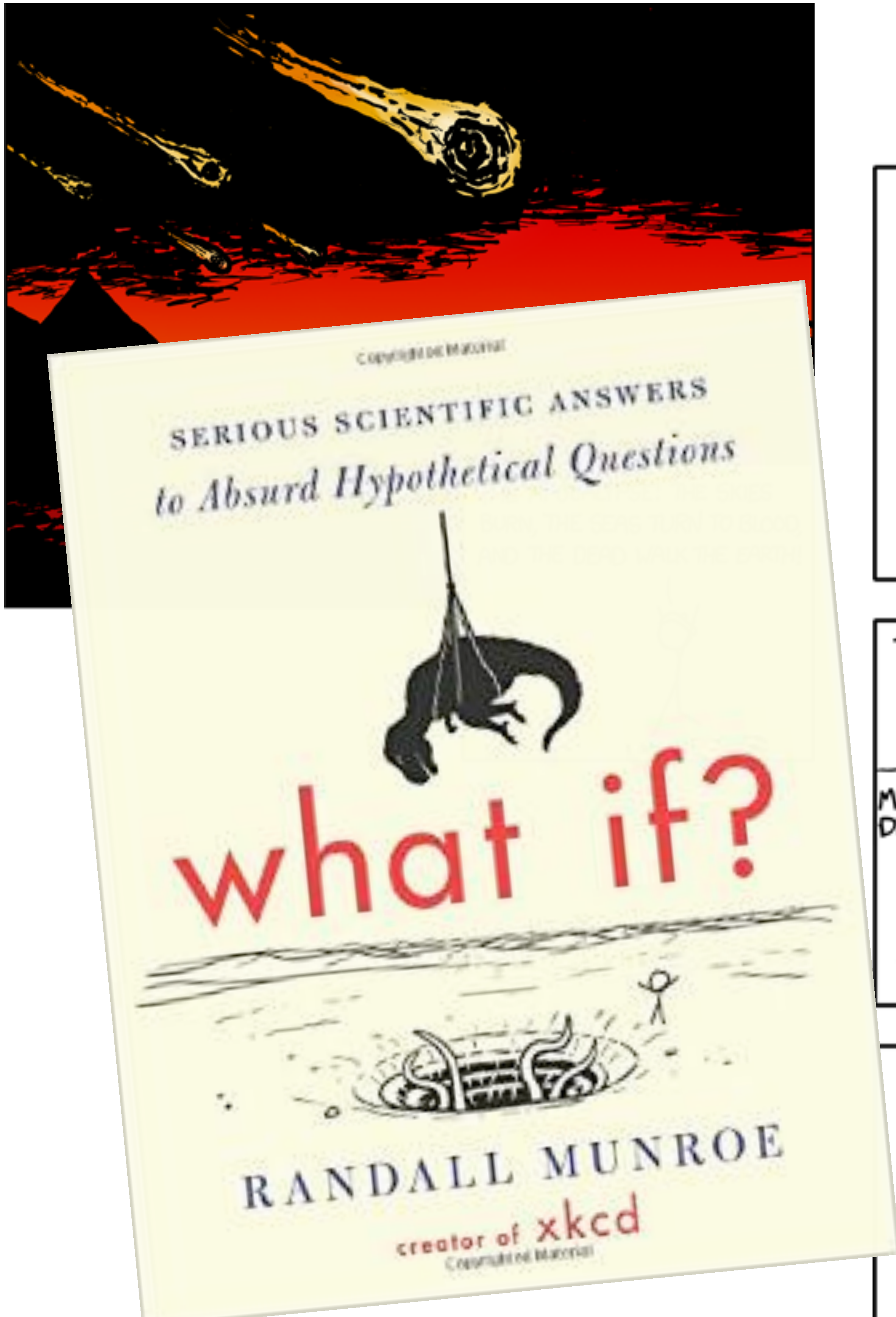


Jaroslav Nešetřil

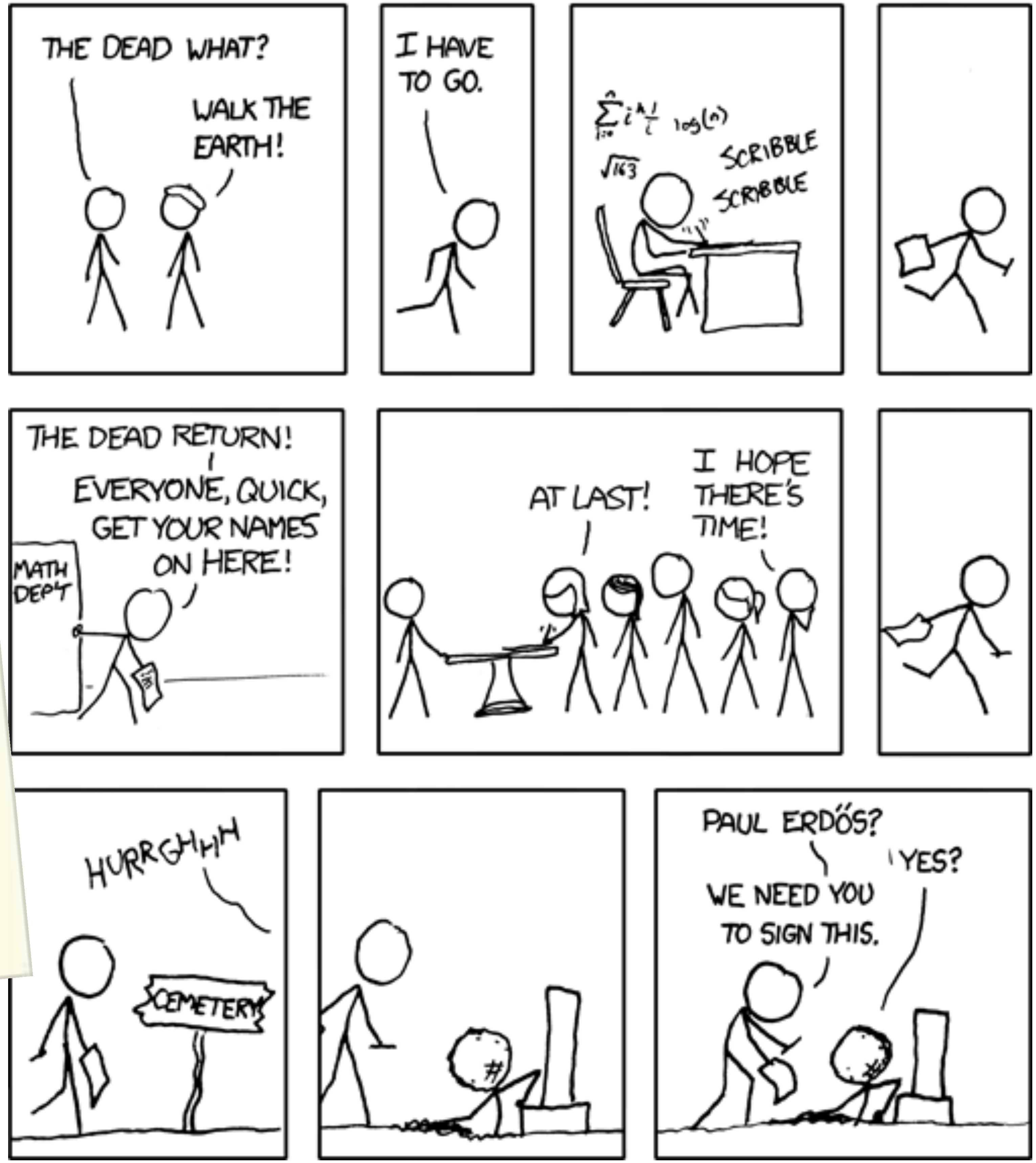
Dieter Kratsch

I have Erdos number 2
(along with 9266 other people)


THE APOCALYPSE! THE SKIES BURN, THE SEAS TURN TO BLOOD, AND THE DEAD WALK THE EARTH!
THE DEAD WHAT?
WALK THE EARTH!
I HAVE TO GO.
SCRIBBLE SCRIBBLE
THE DEAD RETURN!
EVERYONE, QUICK, GET YOUR NAMES ON HERE!
MATH DEP'T
AT LAST!
I HOPE THERE'S TIME!
HURRGHHHH
CEMETERY
PAUL ERDŐS?
YES?
WE NEED YOU TO SIGN THIS.
xkcd


SERIOUS SCIENTIFIC ANSWERS
to Absurd Hypothetical Questions
what if?
RANDALL MUNROE
creator of xkcd

xkcd


THE DEAD WHAT?
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MATH DEP'T
AT LAST!
I HOPE THERE'S TIME!
HURRGHHHH
CEMETERY
PAUL ERDŐS?
YES?
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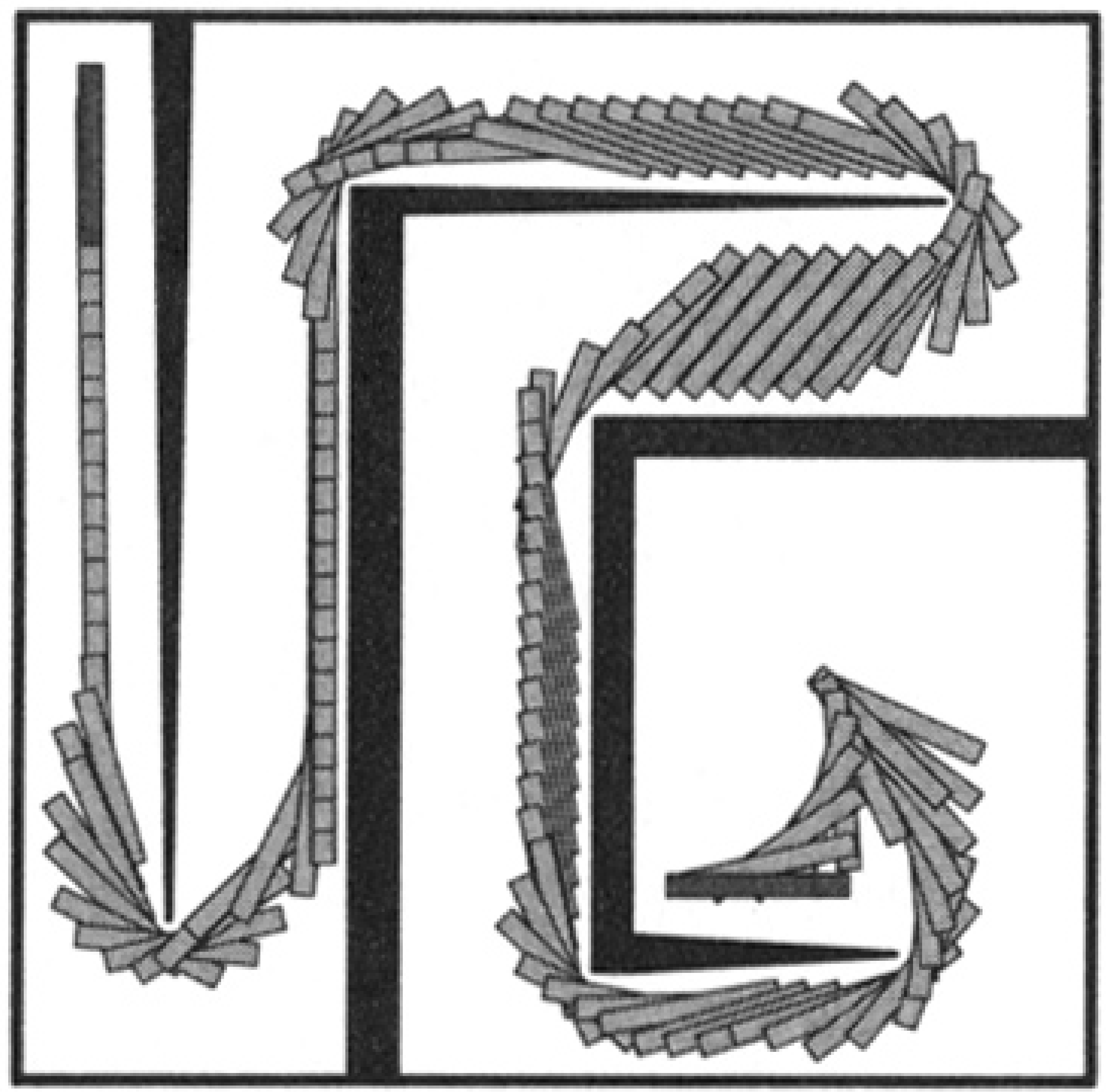
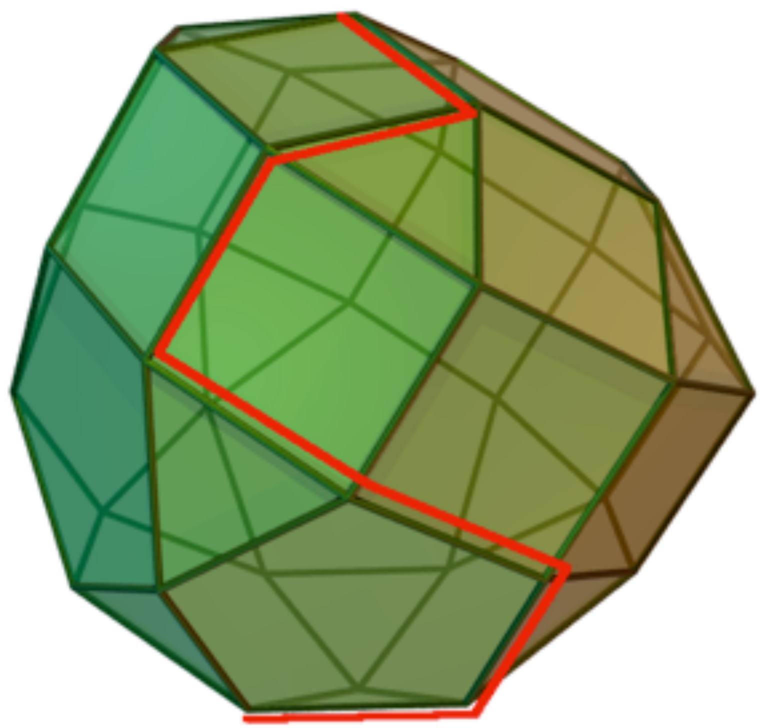
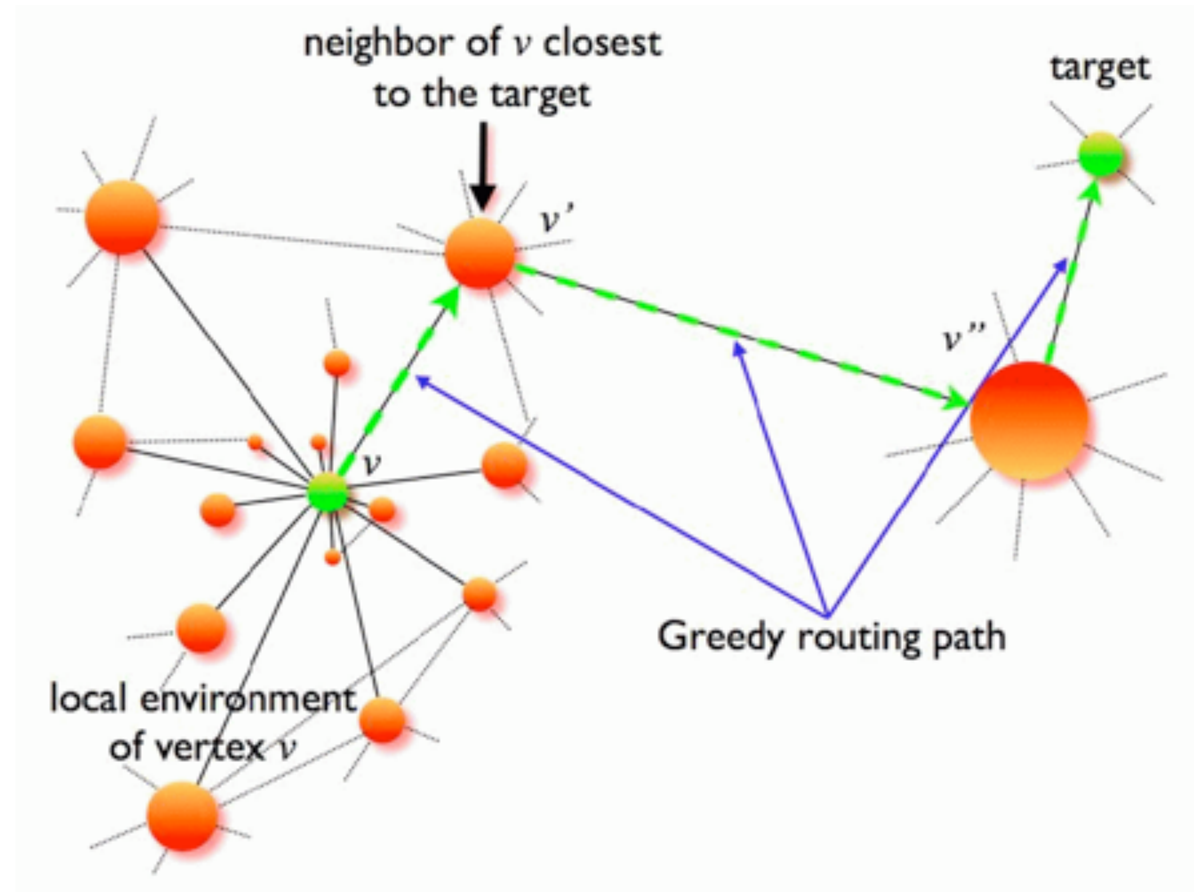
# Motion Planning



# Shortest Paths



linear programming
simplex method




network routing

# Shortest Path Algorithms

- in a graph
- in a geometric space

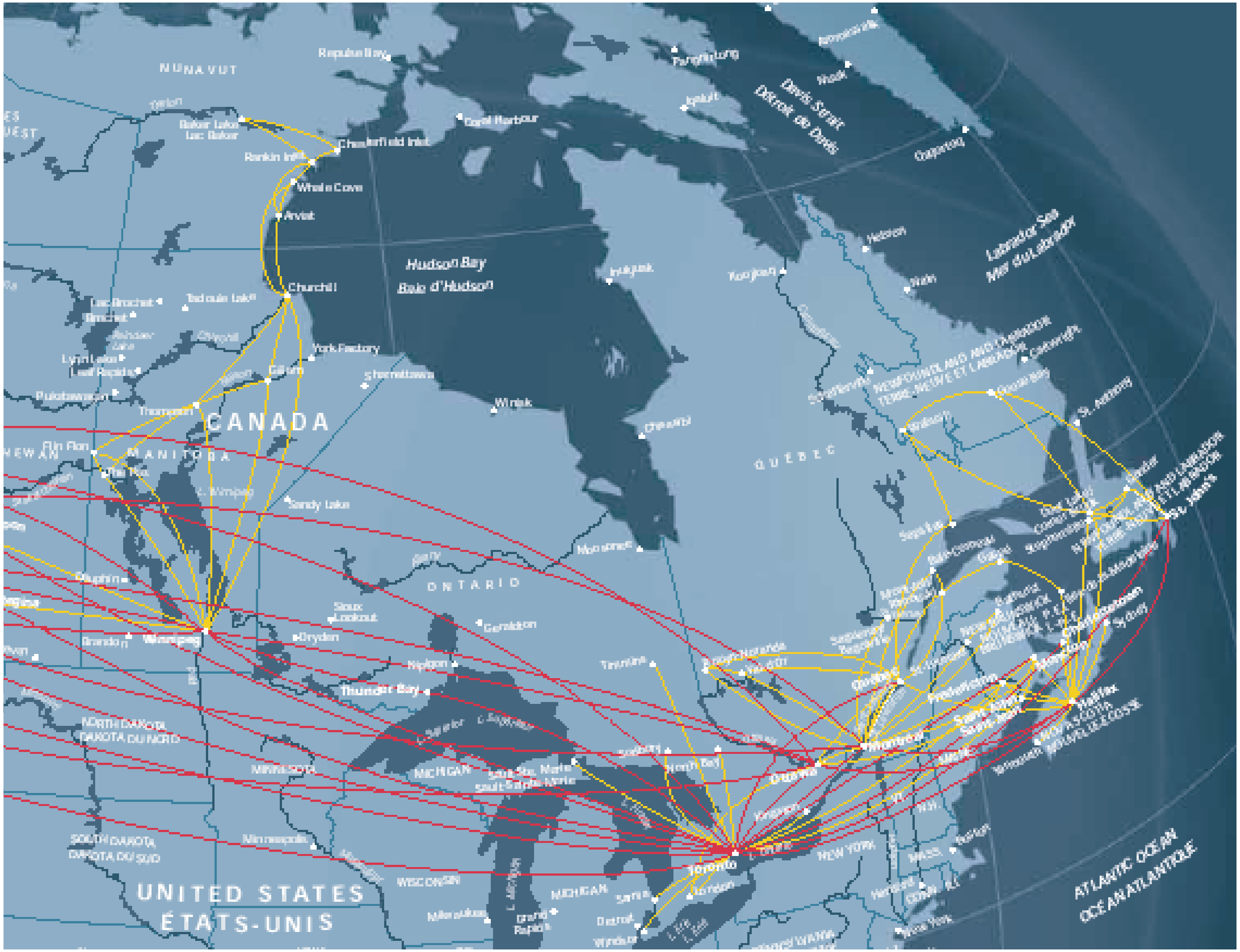
## types of questions

- given start point, end point, find shortest path
- "single source": given start point, find shortest paths to all end points
- "all pairs": find shortest path for all start points, all end points

Also "query" versions.

# Shortest Paths in Graphs

# Shortest Paths in Graphs



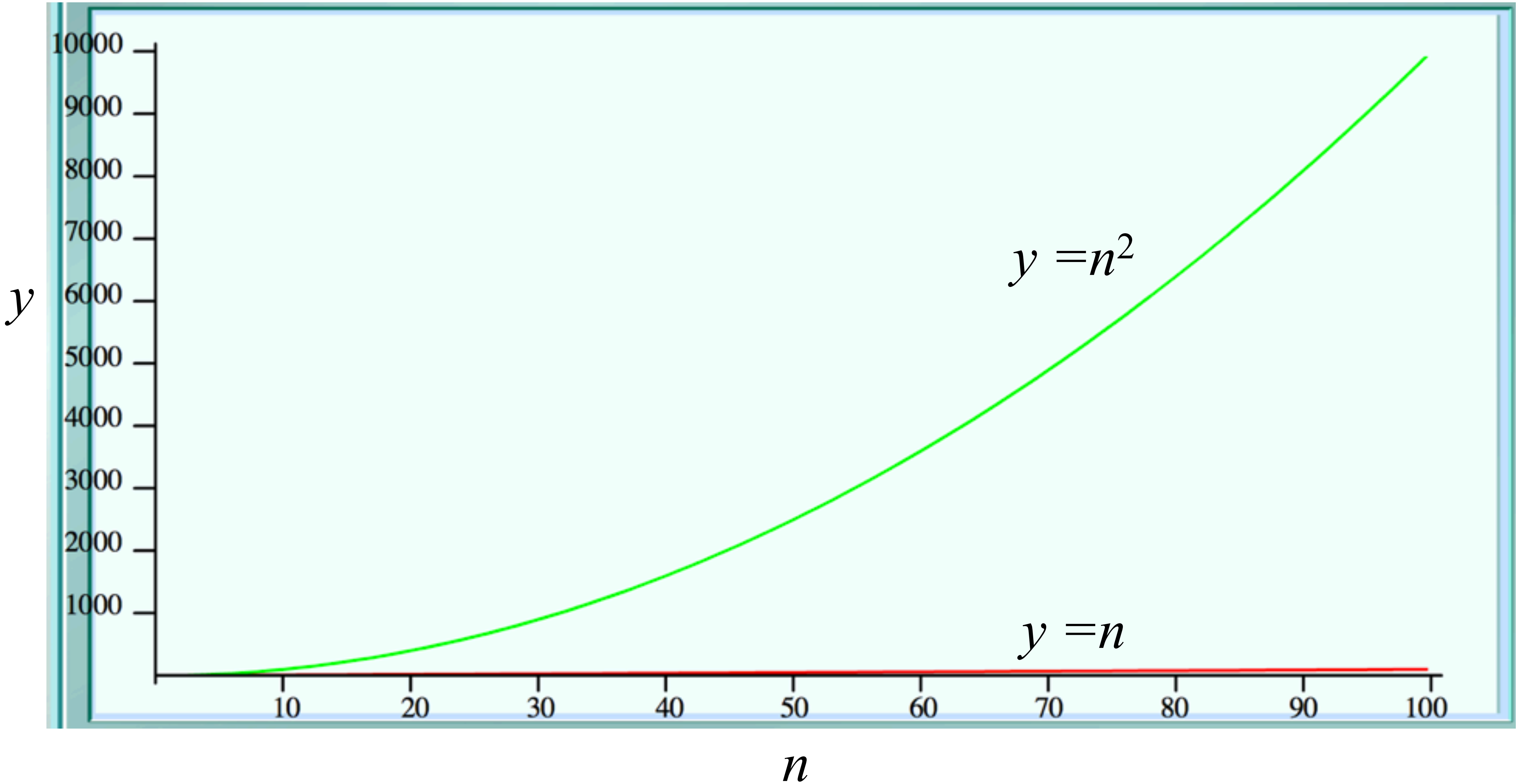
# Finding Shortest Paths in Graphs

# Can we store all the answers?

| Distances in KM * | Banff | Bella Coola | Calgary | Campbell River | Edmonton | Jasper | Kamloops | Kelowna | Nelson | Port Hardy | Prince George | Prince Rupert | Revelstoke | Tofino | Vancouver | Victoria | Whistler |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Banff** | | 1235 | 128 | 1004 | 423 | 285 | 492 | 474 | 496 | 1242 | 661 | 1385 | 282 | 1059 | 847 | 885 | 967 |
| **Bella Coola** | 1235 | | 1363 | 238* | 1360 | 999 | 743 | 903 | 1197 | 0 | 694 | 0 | 953 | 512 | 996 | 504 | 1116 |
| **Calgary** | 128 | 1363 | | 1132 | 295 | 413 | 620 | 602 | 624 | 1370 | 789 | 1513 | 410 | 1187 | 975 | 1013 | 1095 |
| **Campbell River** | 1004 | 238 | 1132 | | 1312 | 951 | 512 | 552 | 814 | 238 | 940 | 238 | 722 | 274 | 172 | 266 | 253 |
| **Edmonton** | 423 | 1360 | 295 | 1312 | | 361 | 800 | 897 | 919 | 1471 | 737 | 1461 | 705 | 1367 | 1155 | 1193 | 1275 |
| **Jasper** | 285 | 999 | 413 | 951 | 361 | | 439 | 602 | 701 | 1110 | 376 | 1100 | 455 | 1006 | 794 | 832 | 914 |
| **Kamloops** | 492 | 743 | 620 | 512 | 800 | 439 | | 163 | 454 | 750 | 525 | 1249 | 210 | 567 | 355 | 393 | 475 |
| **Kelowna** | 474 | 903 | 602 | 552 | 897 | 602 | 163 | | 338 | 790 | 685 | 1409 | 192 | 607 | 395 | 433 | 515 |
| **Nelson** | 496 | 1197 | 624 | 814 | 919 | 701 | 454 | 338 | | 1052 | 979 | 1703 | 246 | 869 | 657 | 695 | 777 |
| **Port Hardy** | 1242 | 0 | 1370 | 238 | 1471 | 1110 | 750 | 790 | 1052 | | 734 | 0 | 960 | 512 | 410 | 504 | 491 |
| **Prince George** | 661 | 694 | 789 | 940 | 737 | 376 | 525 | 685 | 979 | 734 | | 724 | 735 | 995 | 778 | 821 | 898 |
| **Prince Rupert** | 1385 | 0 | 1513 | 238 | 1461 | 1100 | 1249 | 1409 | 1703 | 0 | 724 | | 1459 | 512 | 1502 | 504 | 1622 |
| **Revelstoke** | 282 | 953 | 410 | 722 | 705 | 455 | 210 | 192 | 246 | 960 | 735 | 1459 | | 777 | 565 | 603 | 685 |
| **Tofino** | 1059 | 512 | 1187 | 274 | 1367 | 1006 | 567 | 607 | 869 | 512 | 995 | 512 | 777 | | 227 | 321 | 308 |
| **Vancouver** | 847 | 996 | 975 | 172 | 1155 | 794 | 355 | 395 | 657 | 410 | 778 | 1502 | 565 | 227 | | 69 | 123 |
| **Victoria** | 885 | 504* | 1013 | 266 | 1193 | 832 | 393 | 433 | 695 | 504 | 821 | 504 | 603 | 321 | 69 | | 192 |
| **Whistler** | 967 | 1116 | 1095 | 253 | 1275 | 914 | 475 | 515 | 777 | 491 | 898 | 1622 | 685 | 308 | 123 | 192 | |

$17 \times 17 = 289$ $\qquad$ $n \times n = n^2$

# Quadratic Growth



# How to Find Shortest Paths

Try all paths and see which is shortest?

How many shortest paths are there?

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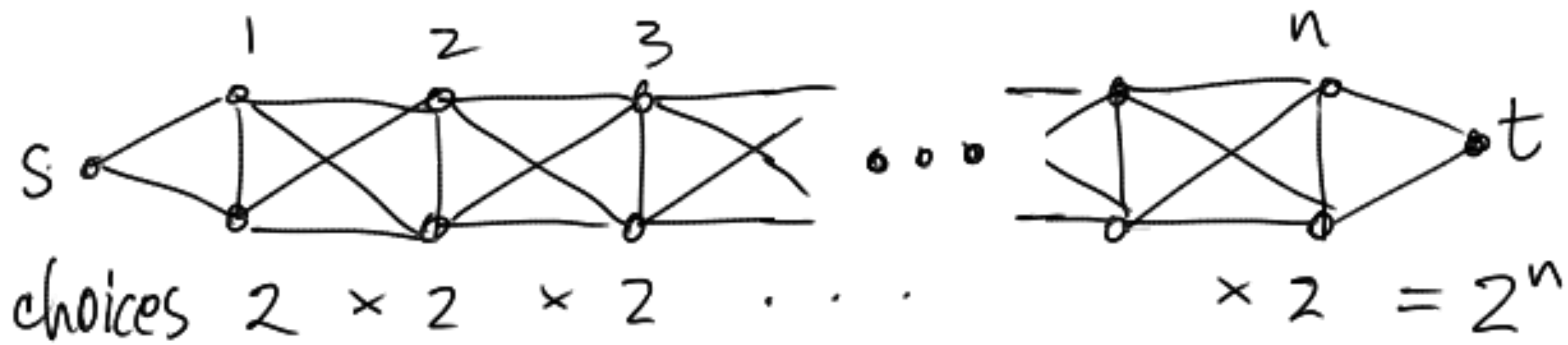
Try all paths and see which is shortest?

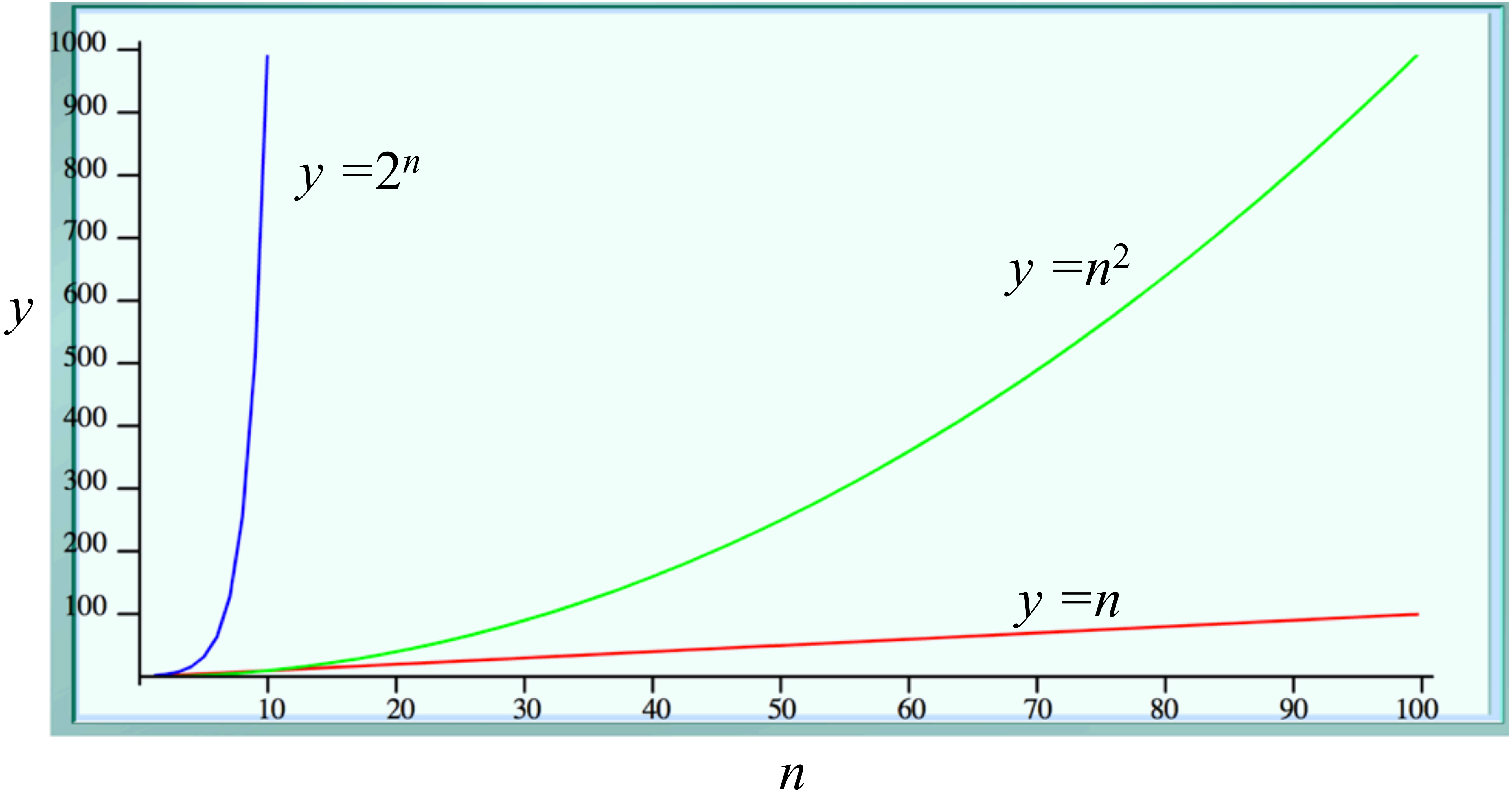
How many shortest paths are there?

graph with 2n+2 vertices



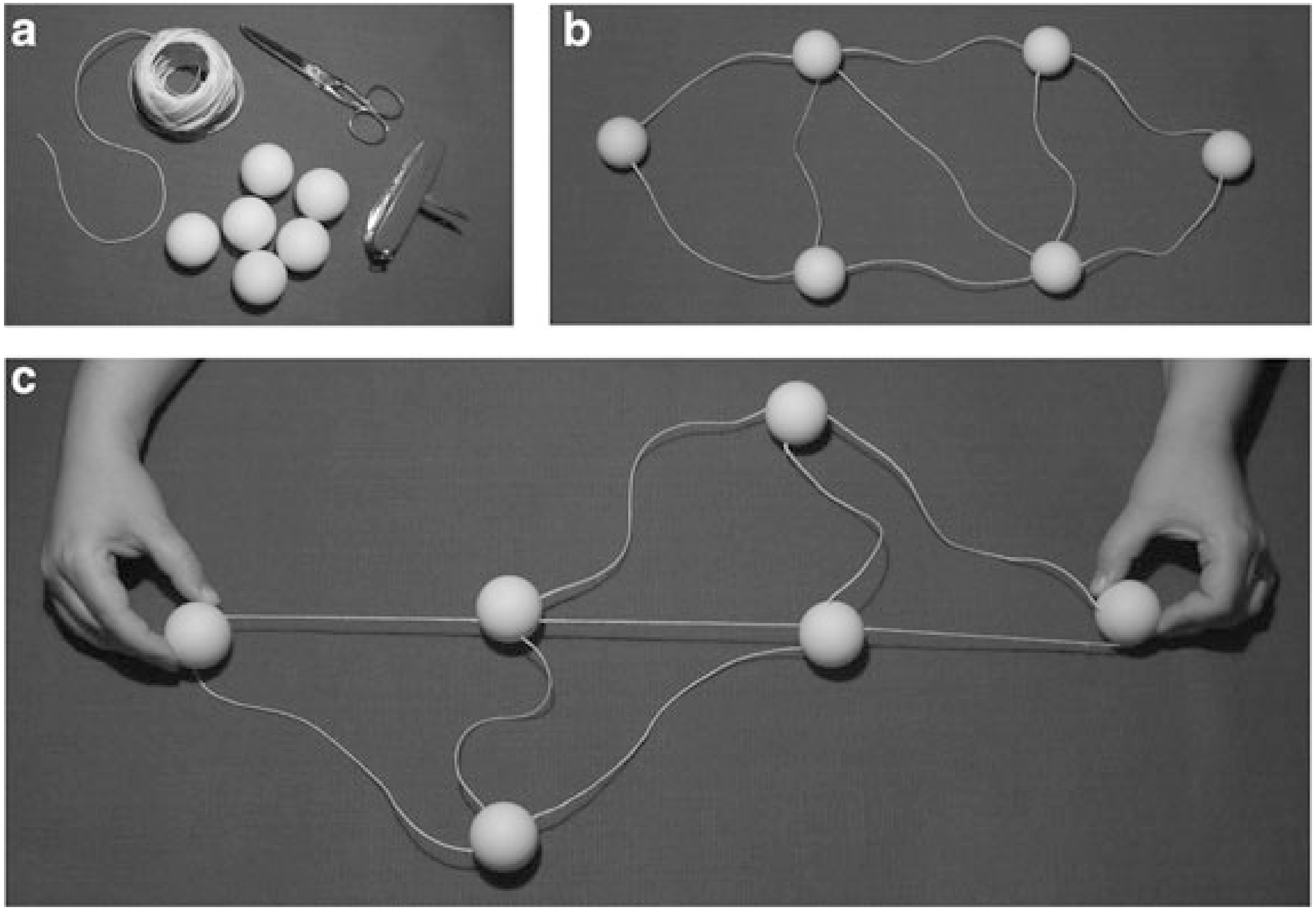

$2^n$ paths

# Exponential Growth

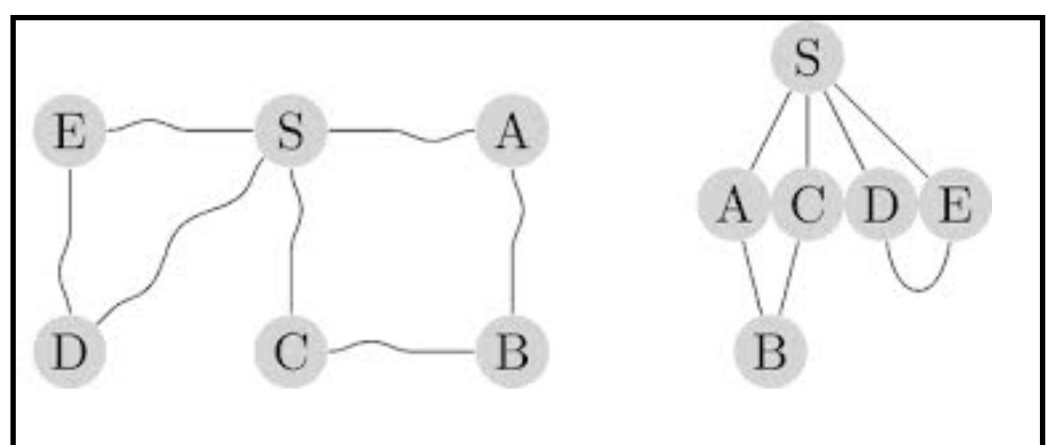



exponential $O(2^n)$ polynomial $O(n^2)$

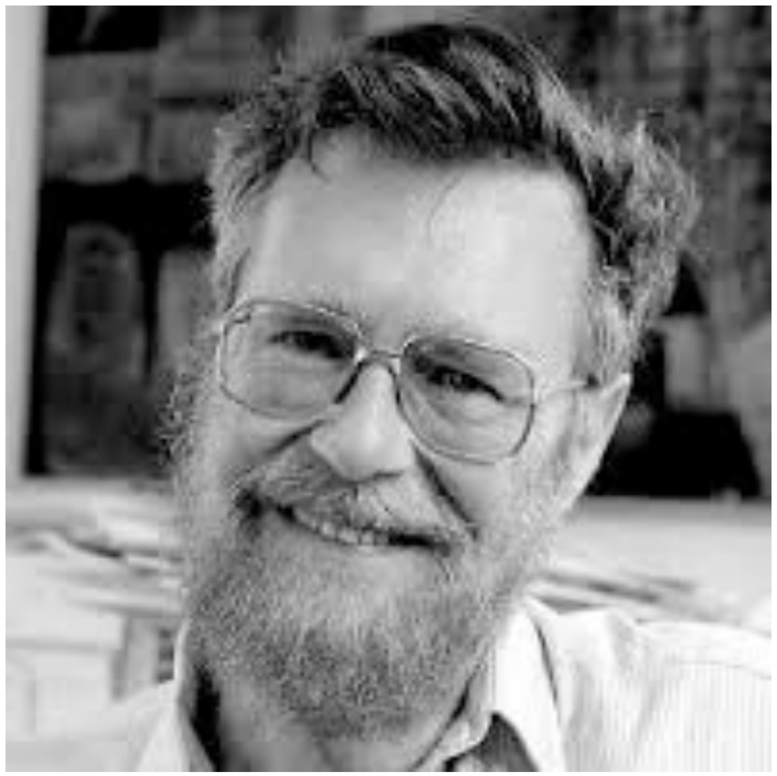
# Shortest Paths – String Computer




single source
shortest paths



# Dijkstra's Algorithm, 1959

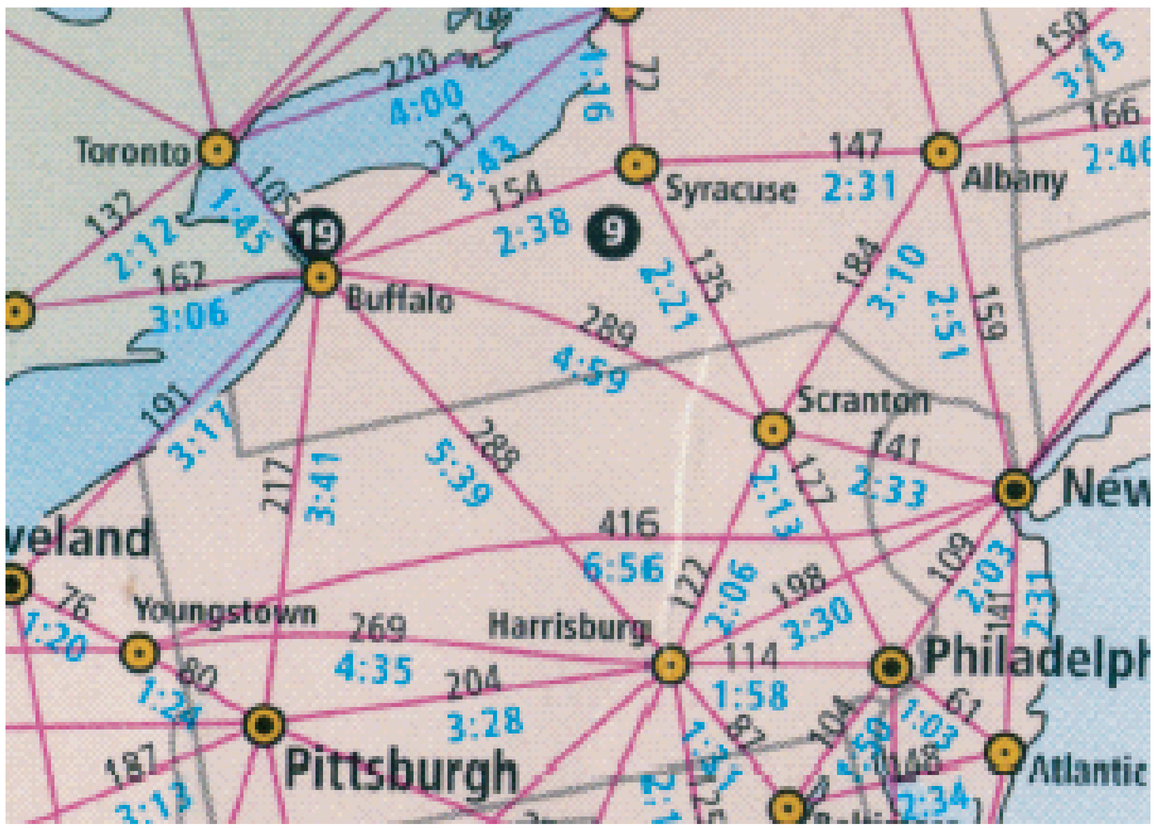


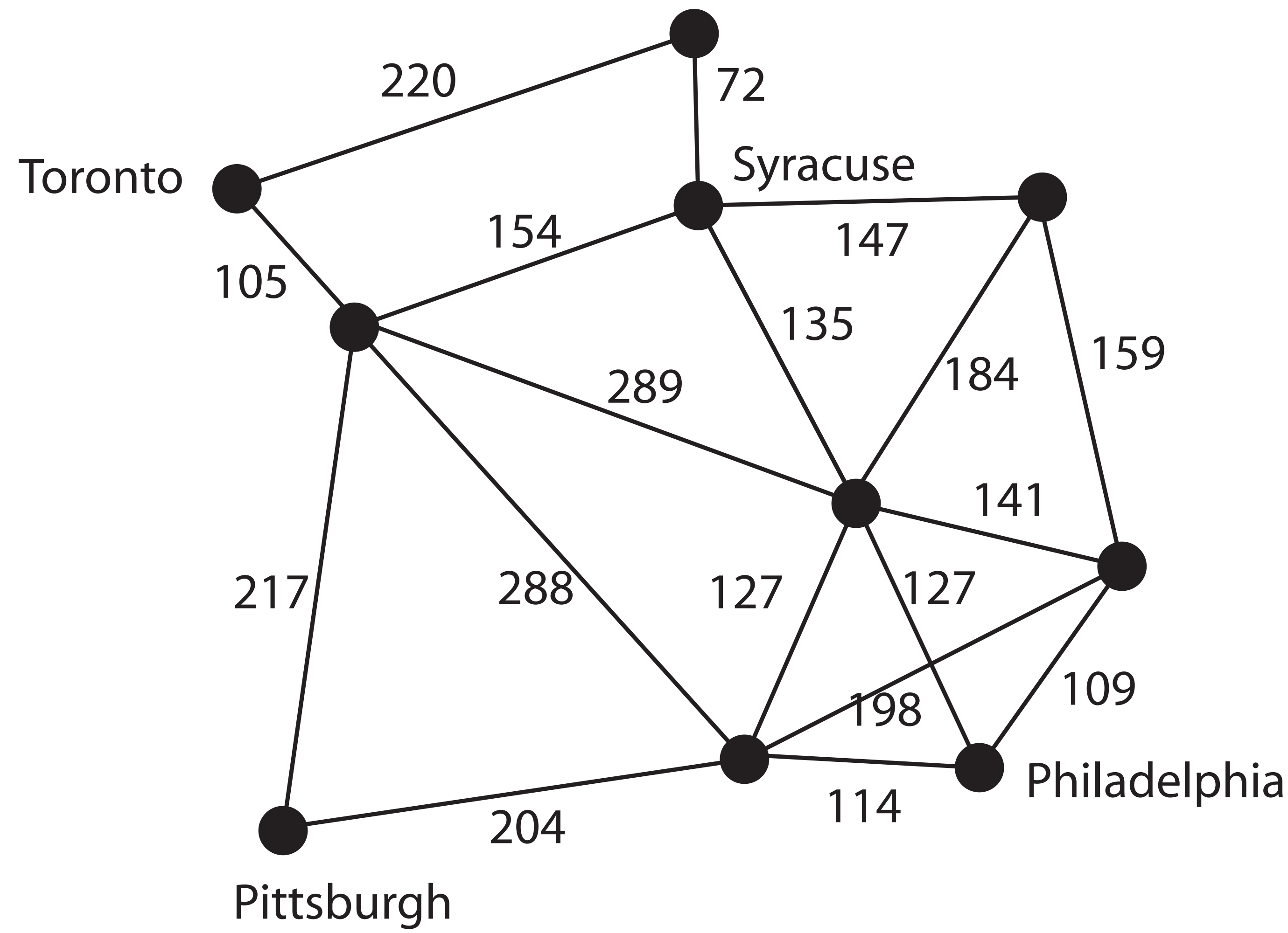
Edsger W. Dijkstra

The first challenge for computing science is to discover how to maintain order in a finite, but very large, discrete universe that is intricately intertwined.

The question of whether Machines Can Think is about as relevant as the question of whether Submarines Can Swim.

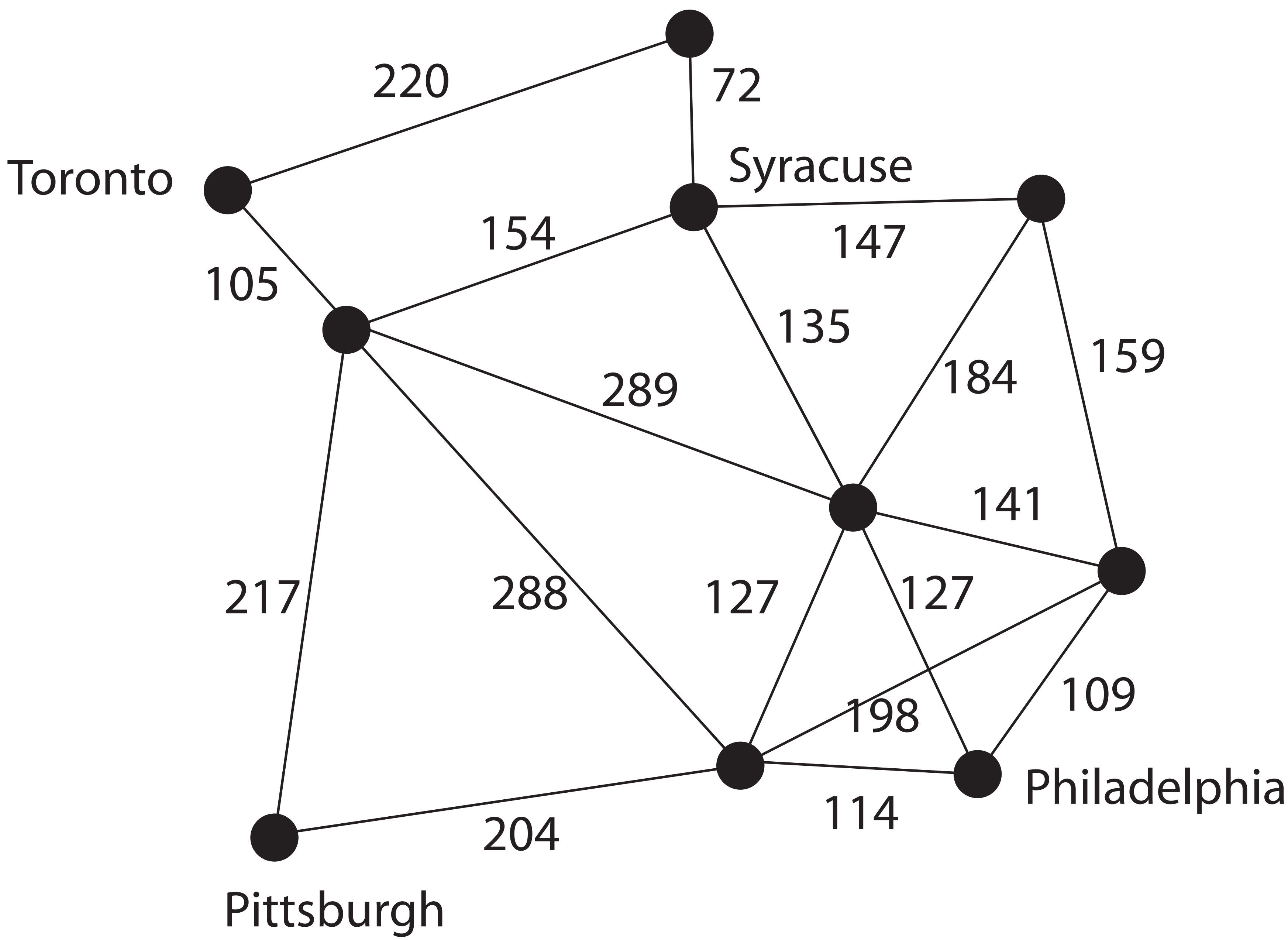
# Dijkstra's Algorithm




Find the shortest path from Toronto to Philadelphia.

# Dijkstra's Algorithm



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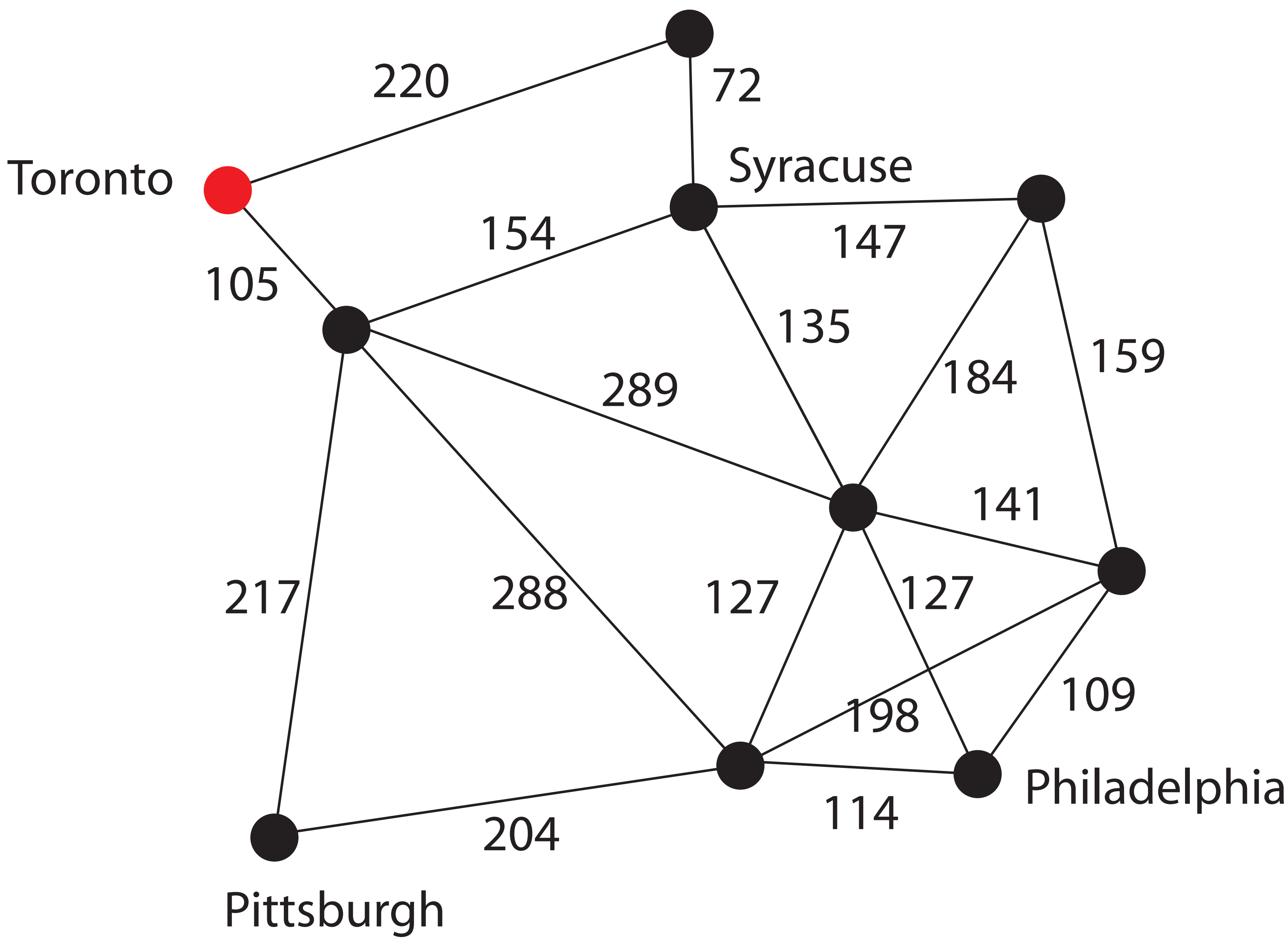

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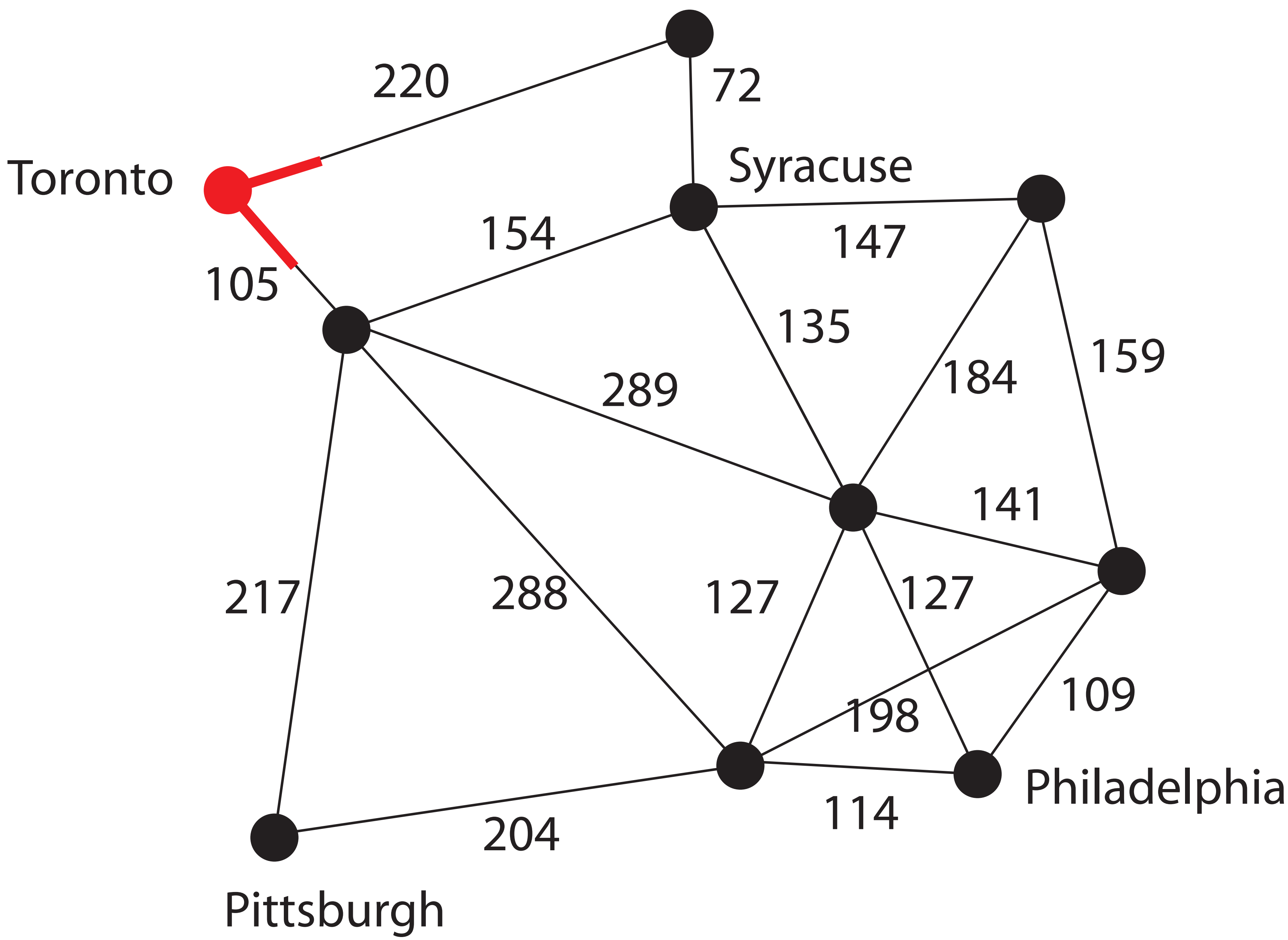

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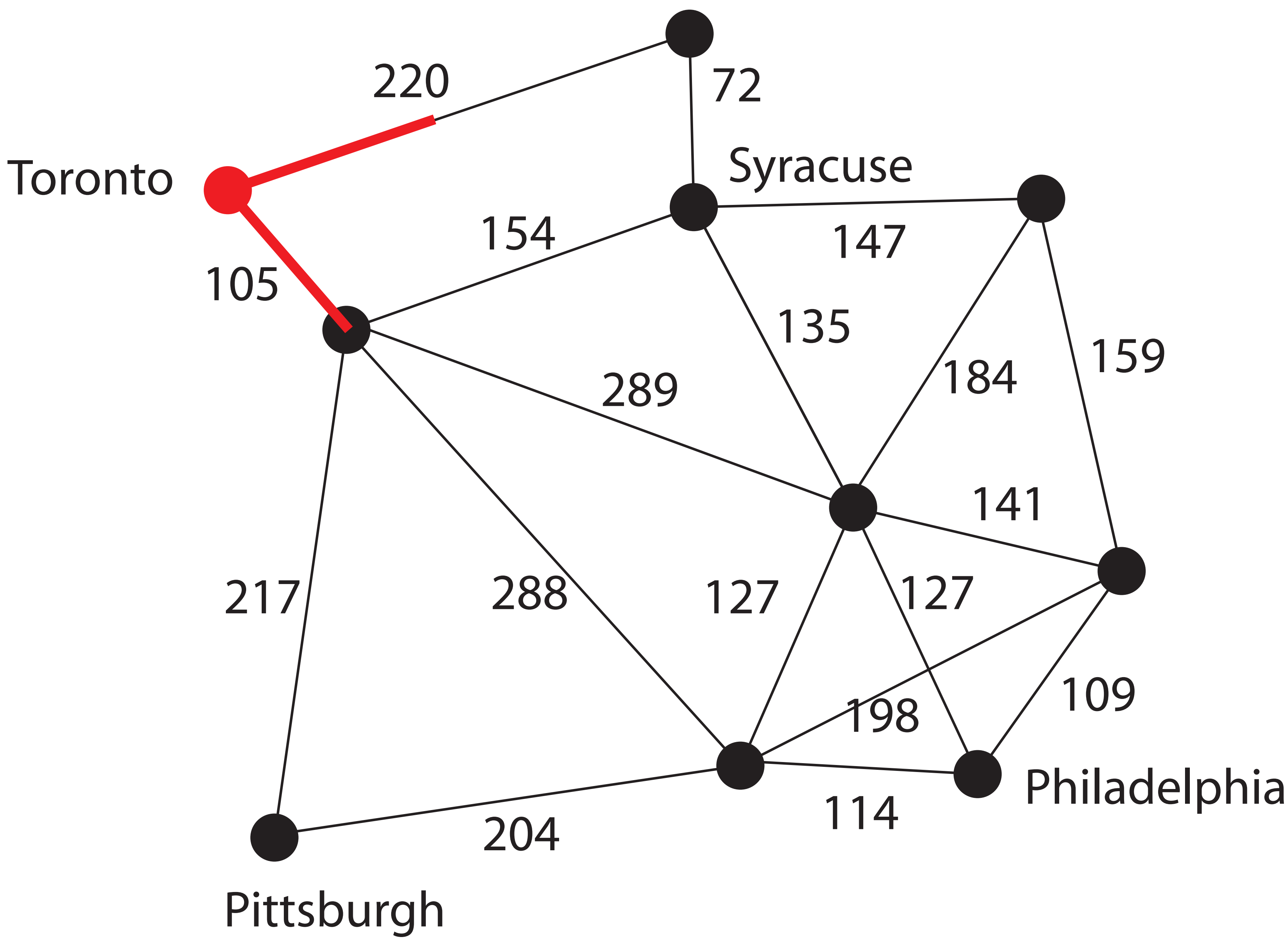

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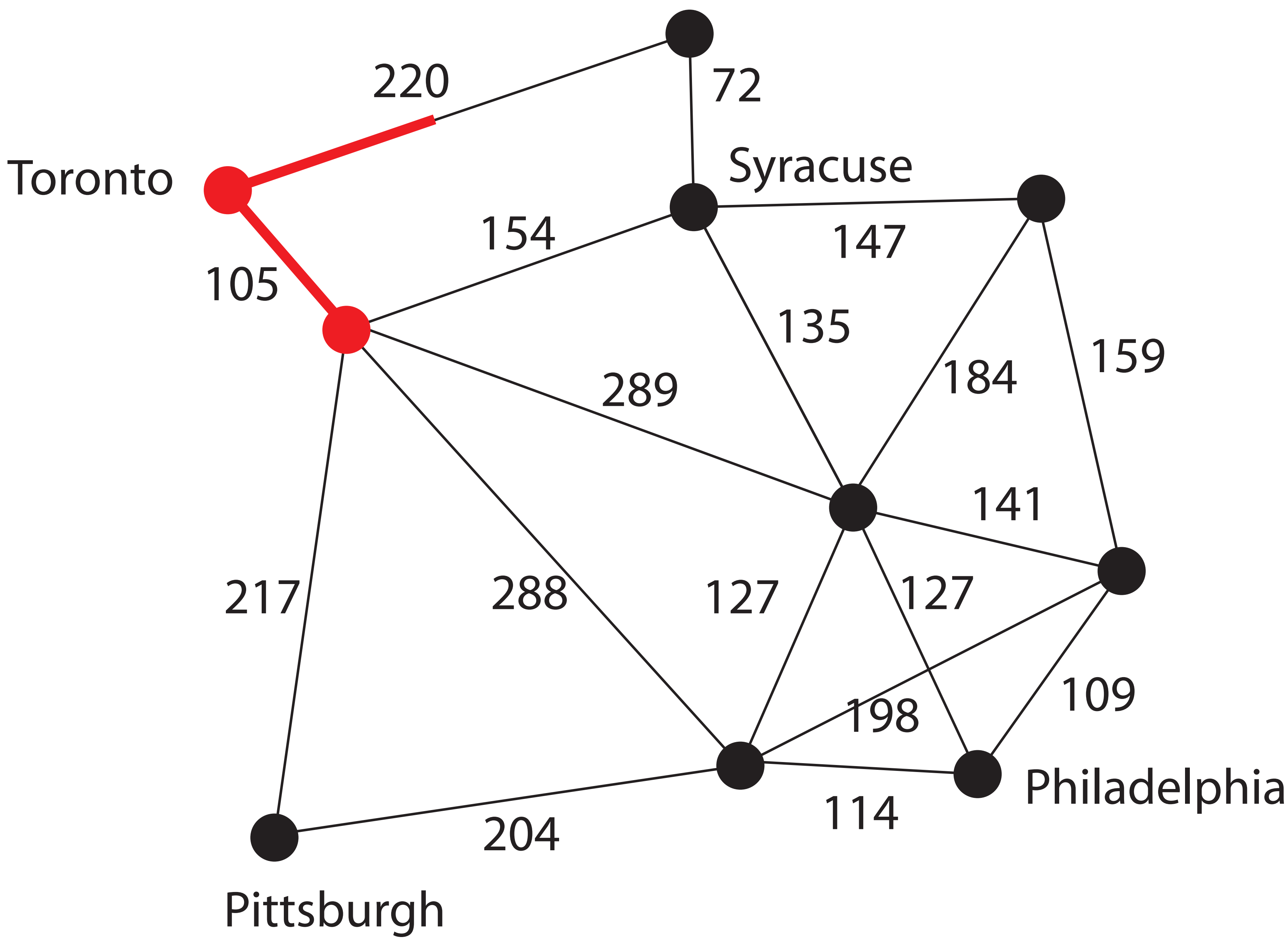

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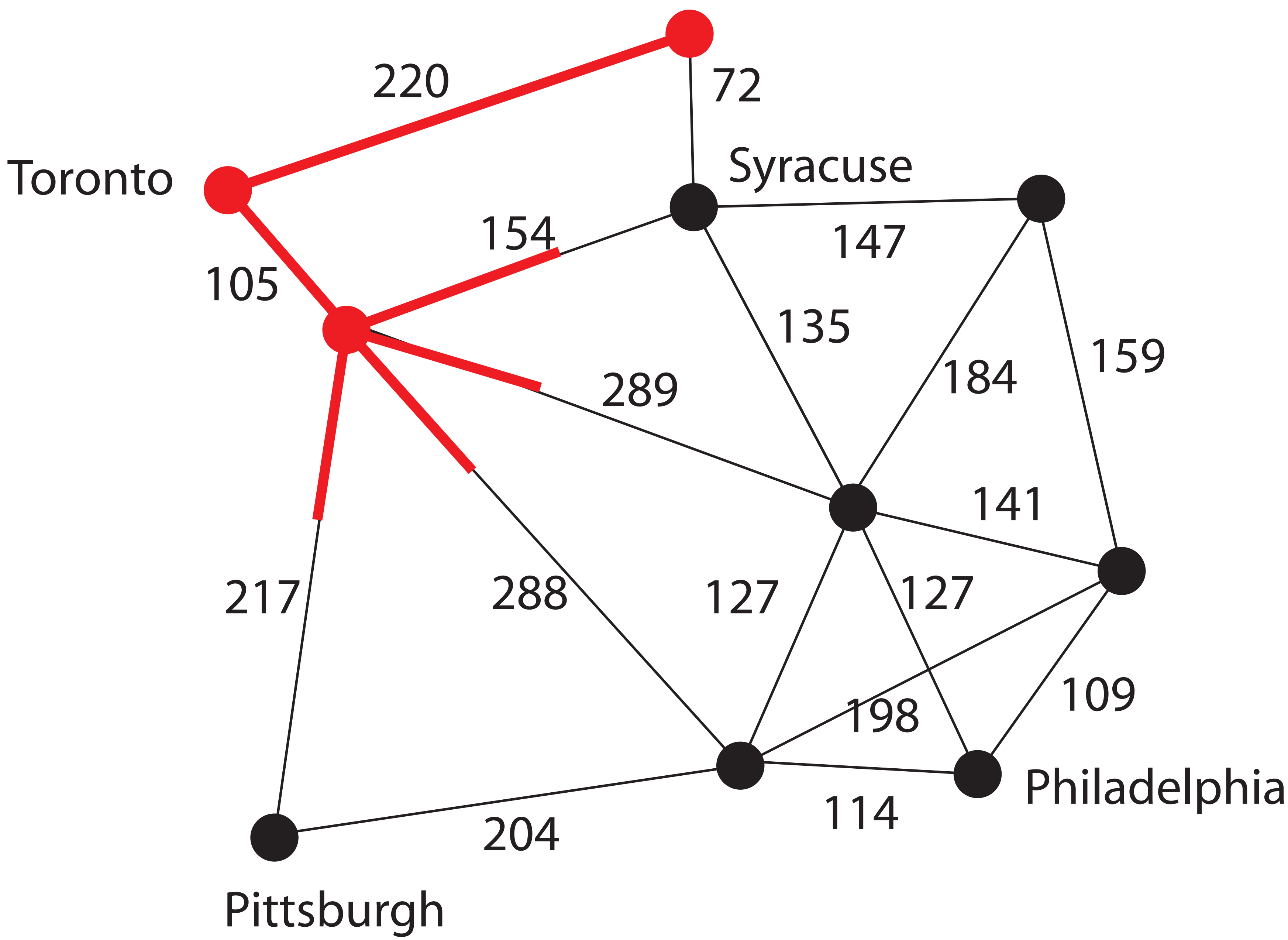

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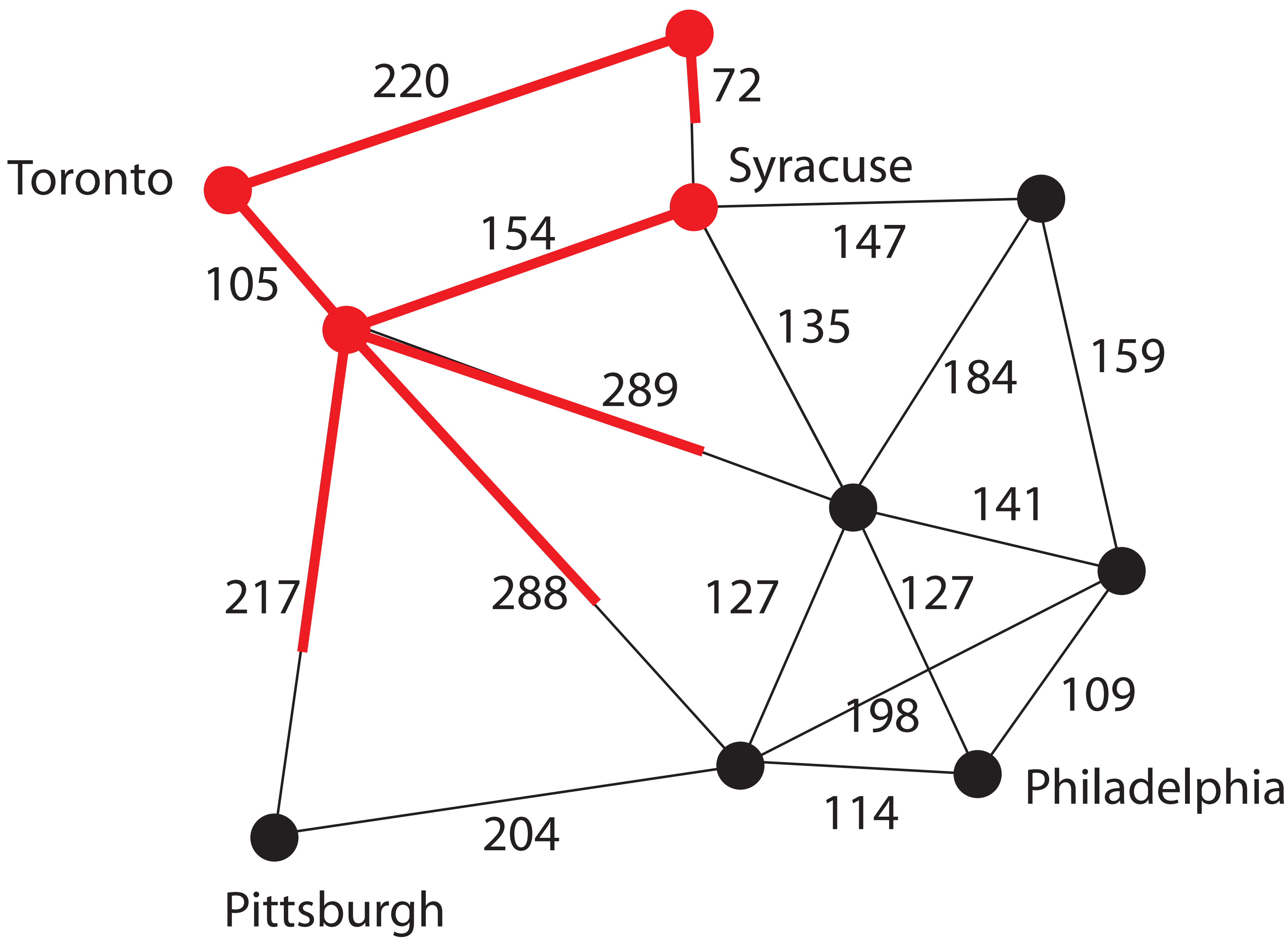

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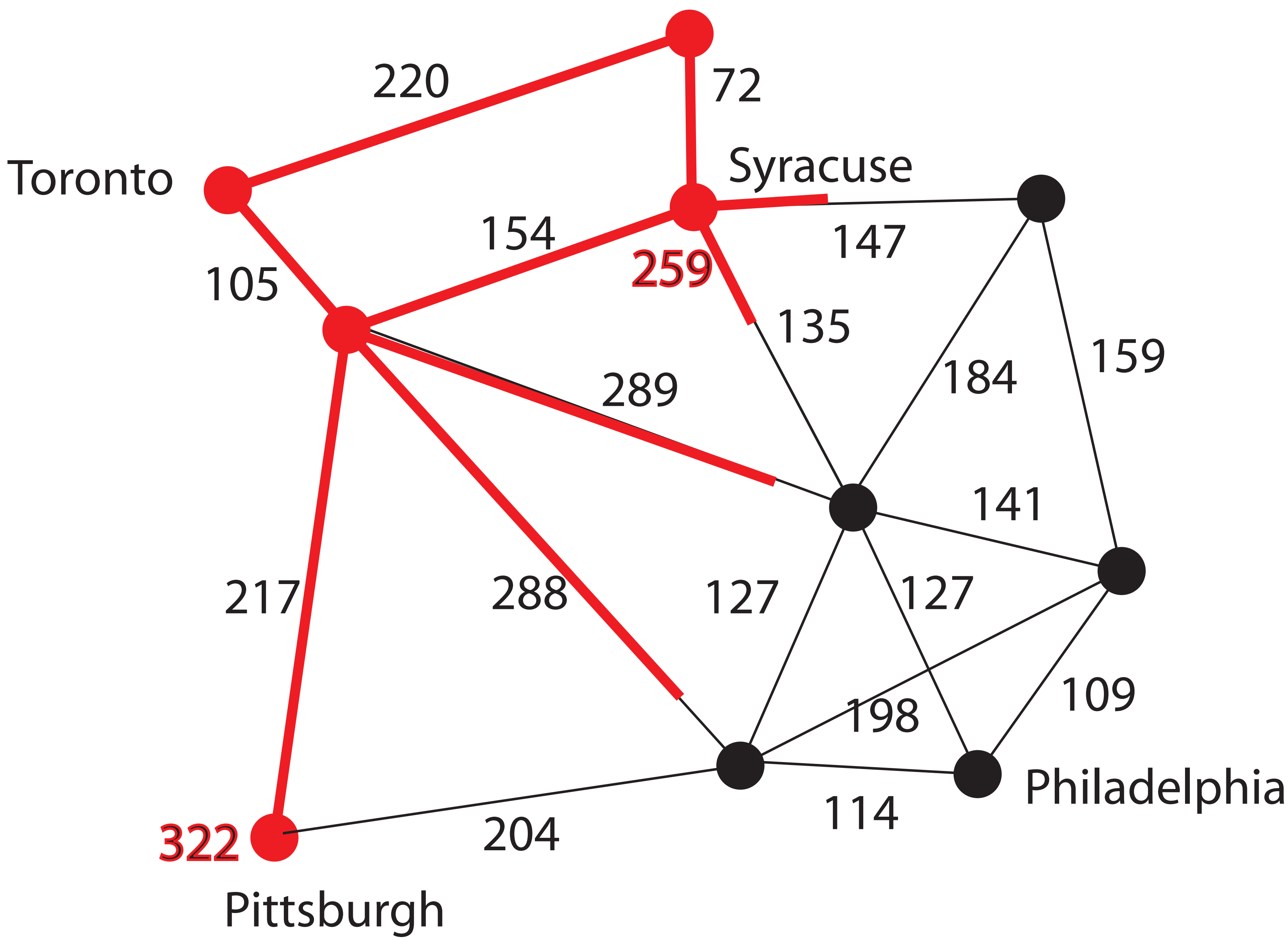

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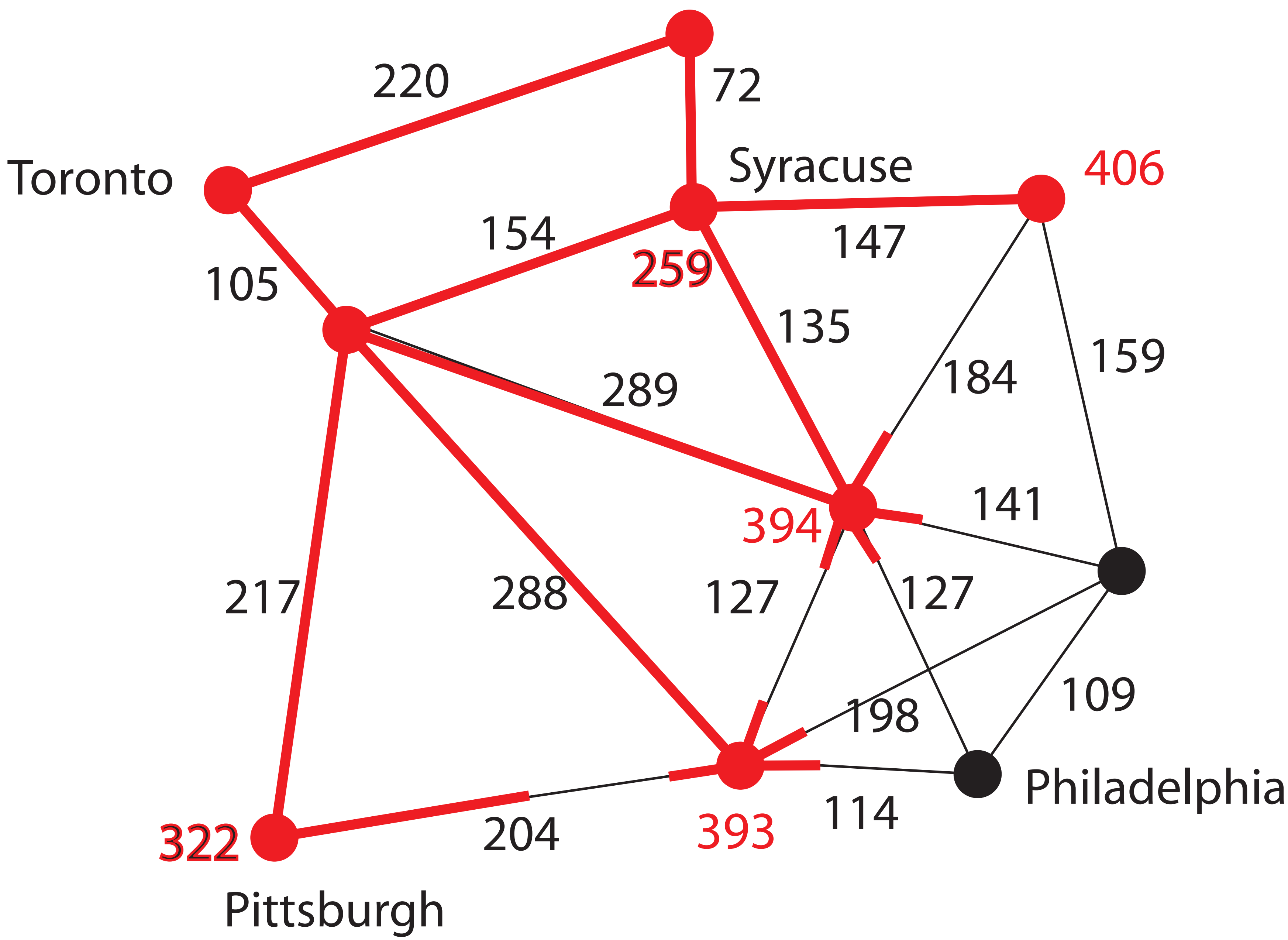

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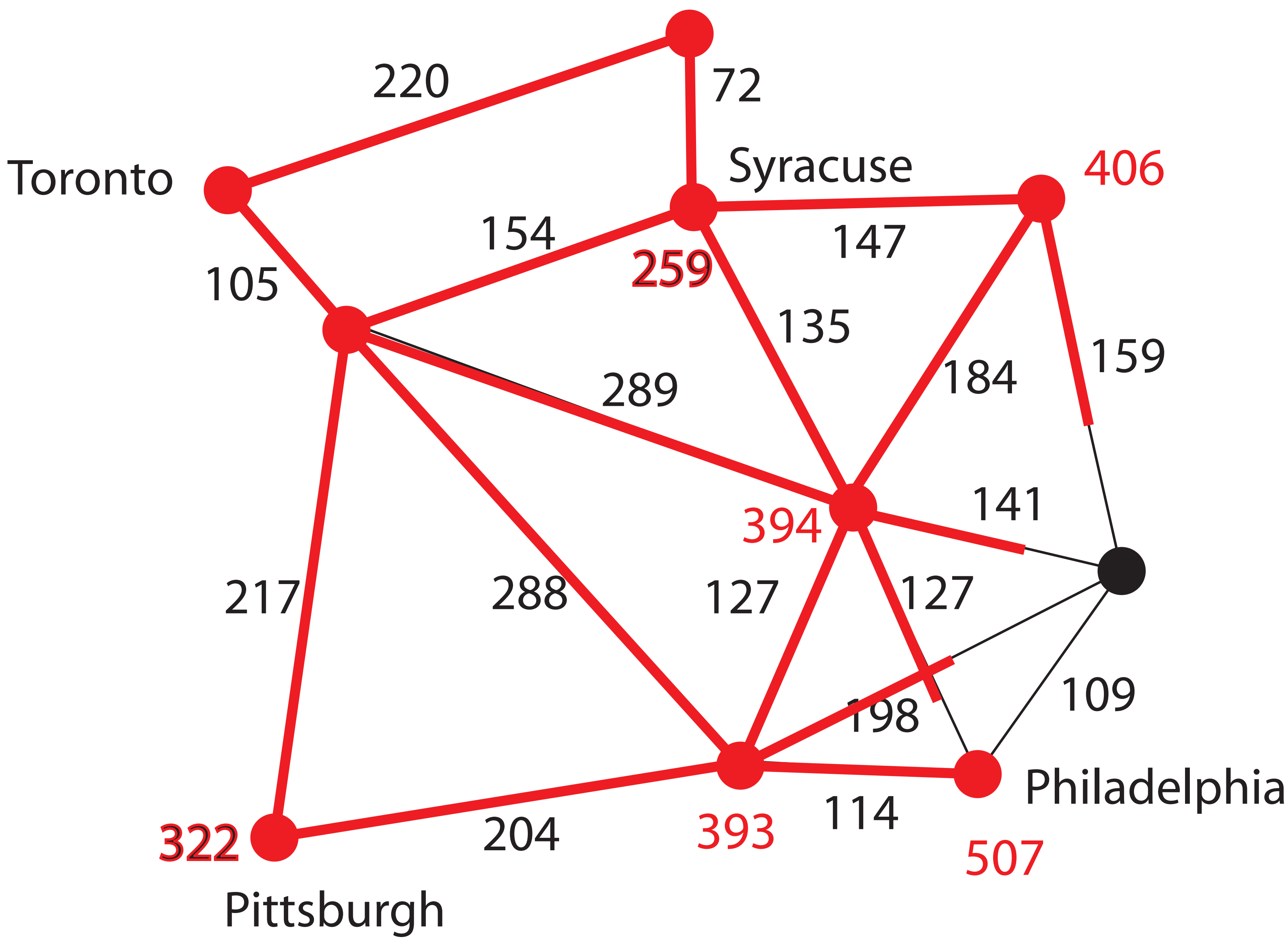

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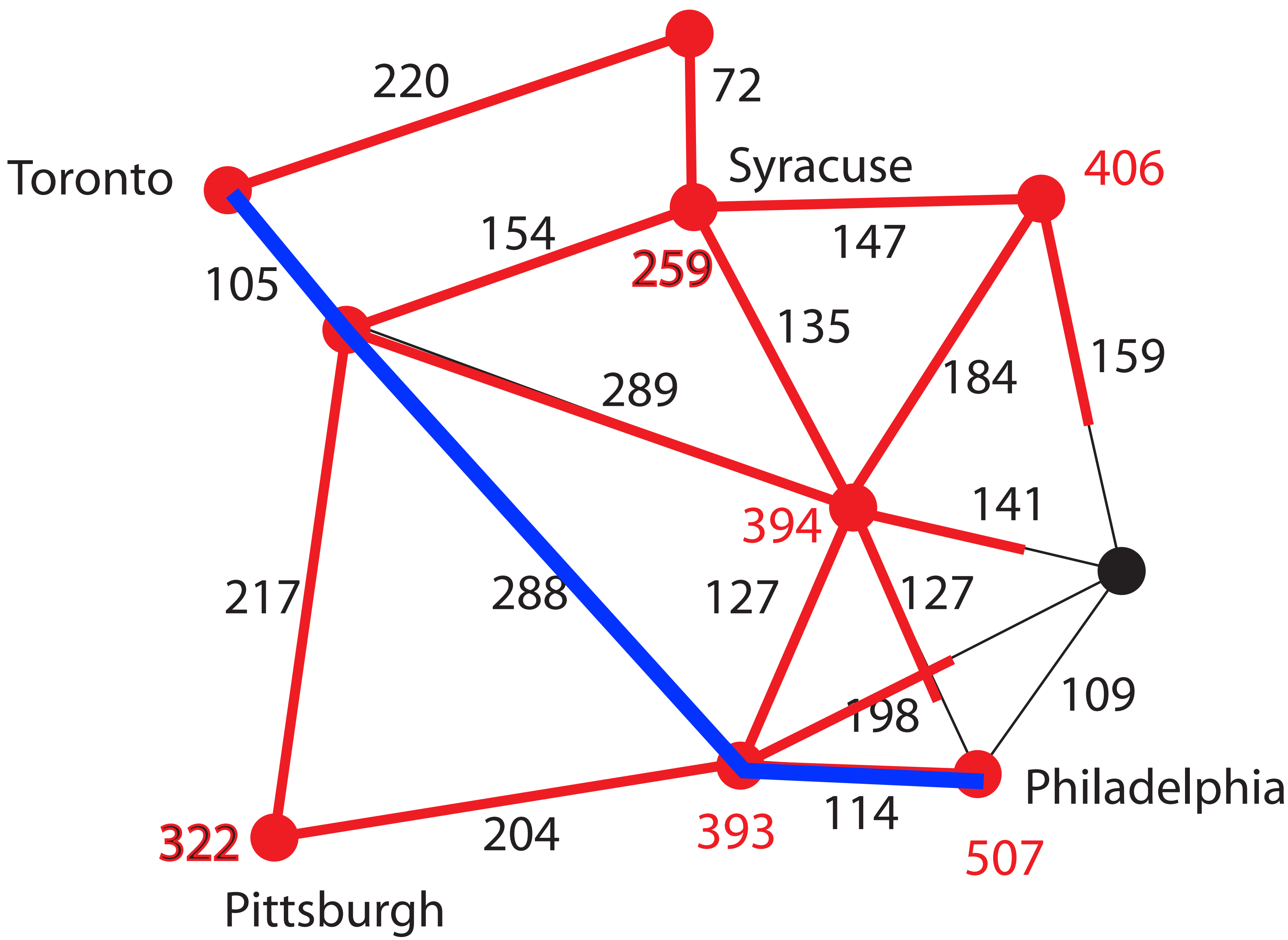


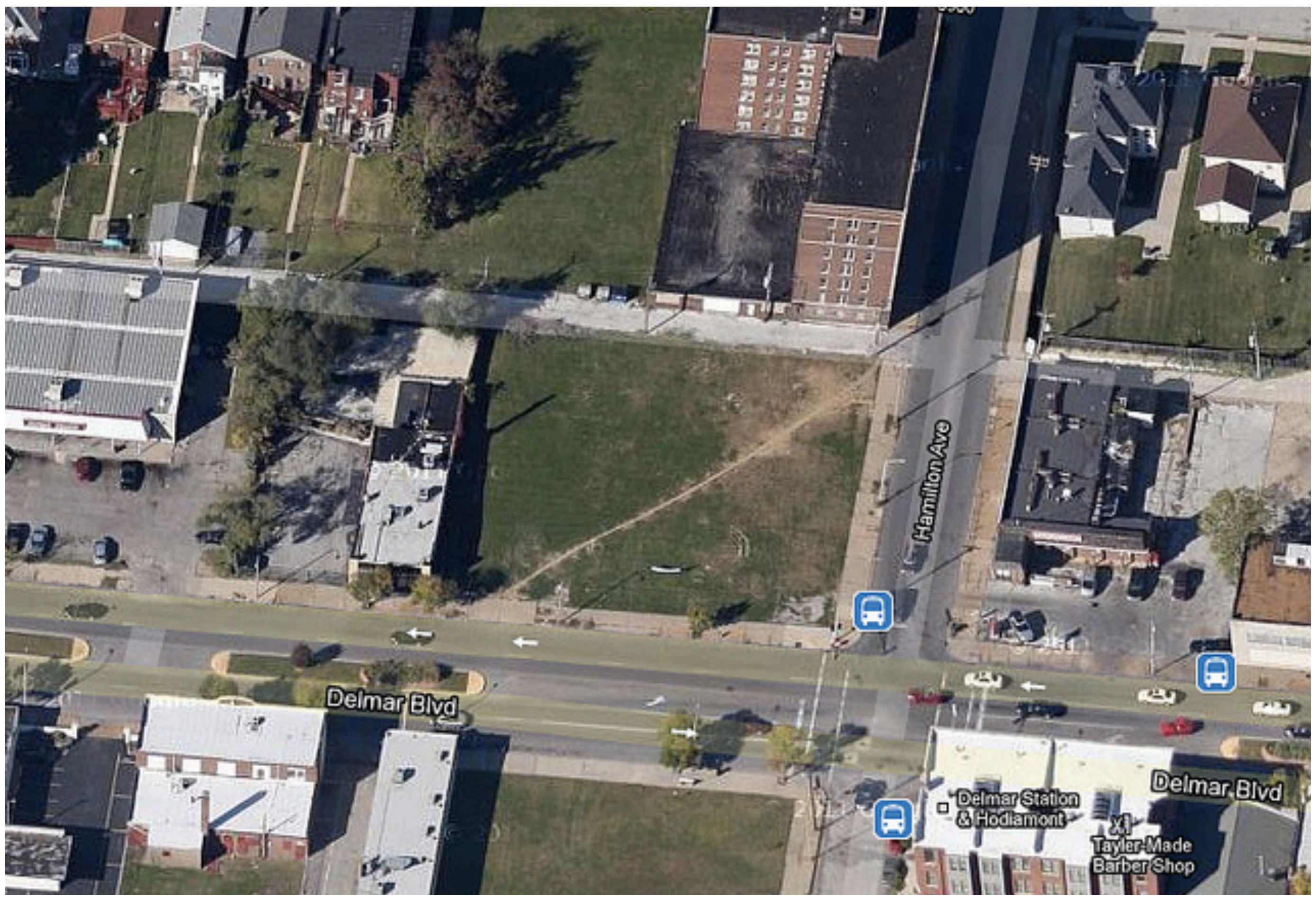
# Dijkstra's Algorithm

running time $O(m + n \log n)$ using Fibonacci heaps

(Fredmand and Tarjan, 1987)

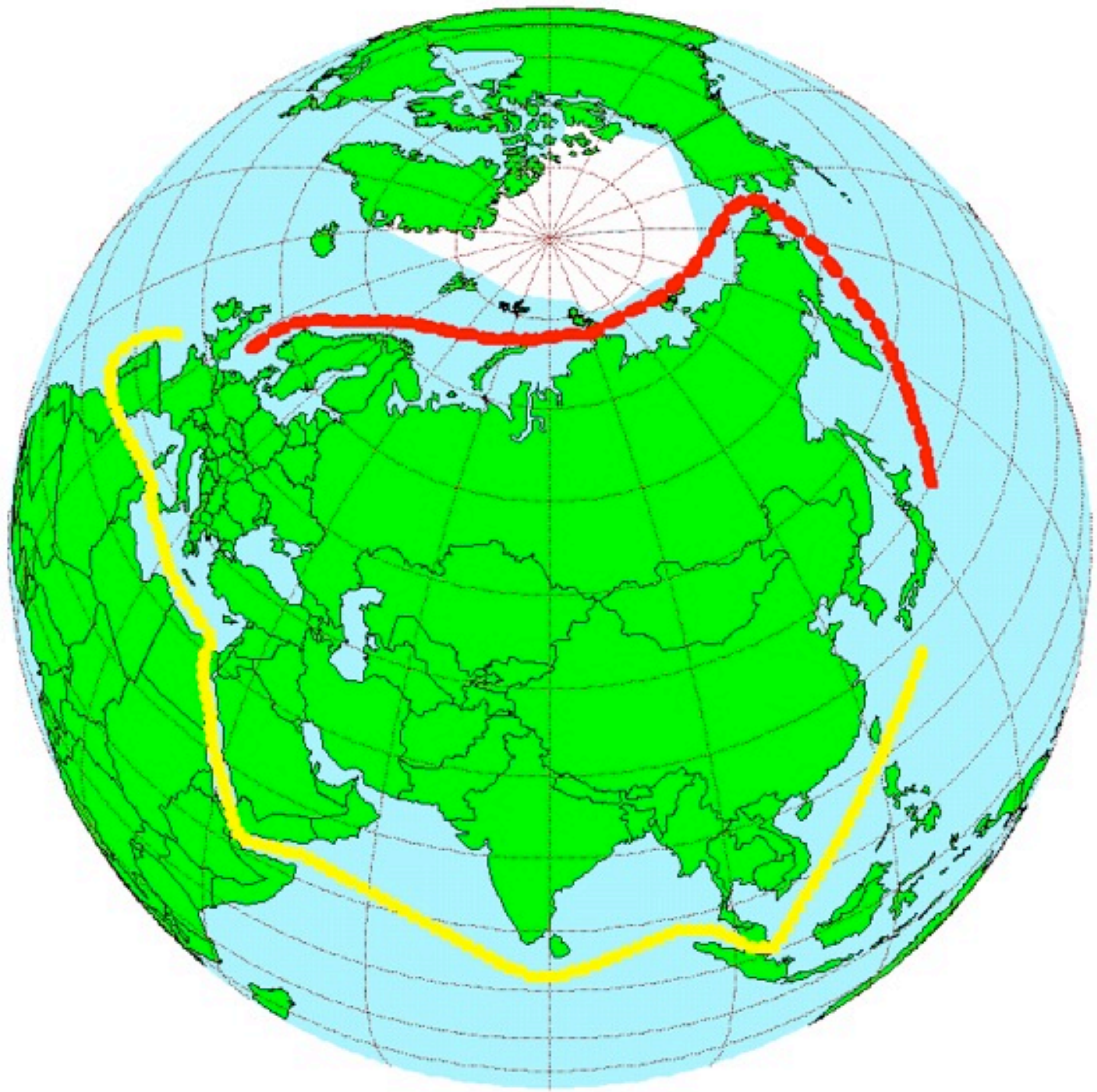
$m$ = number of edges
$n$ = number of vertices

# Paths through Space




desire paths

# Paths through Space



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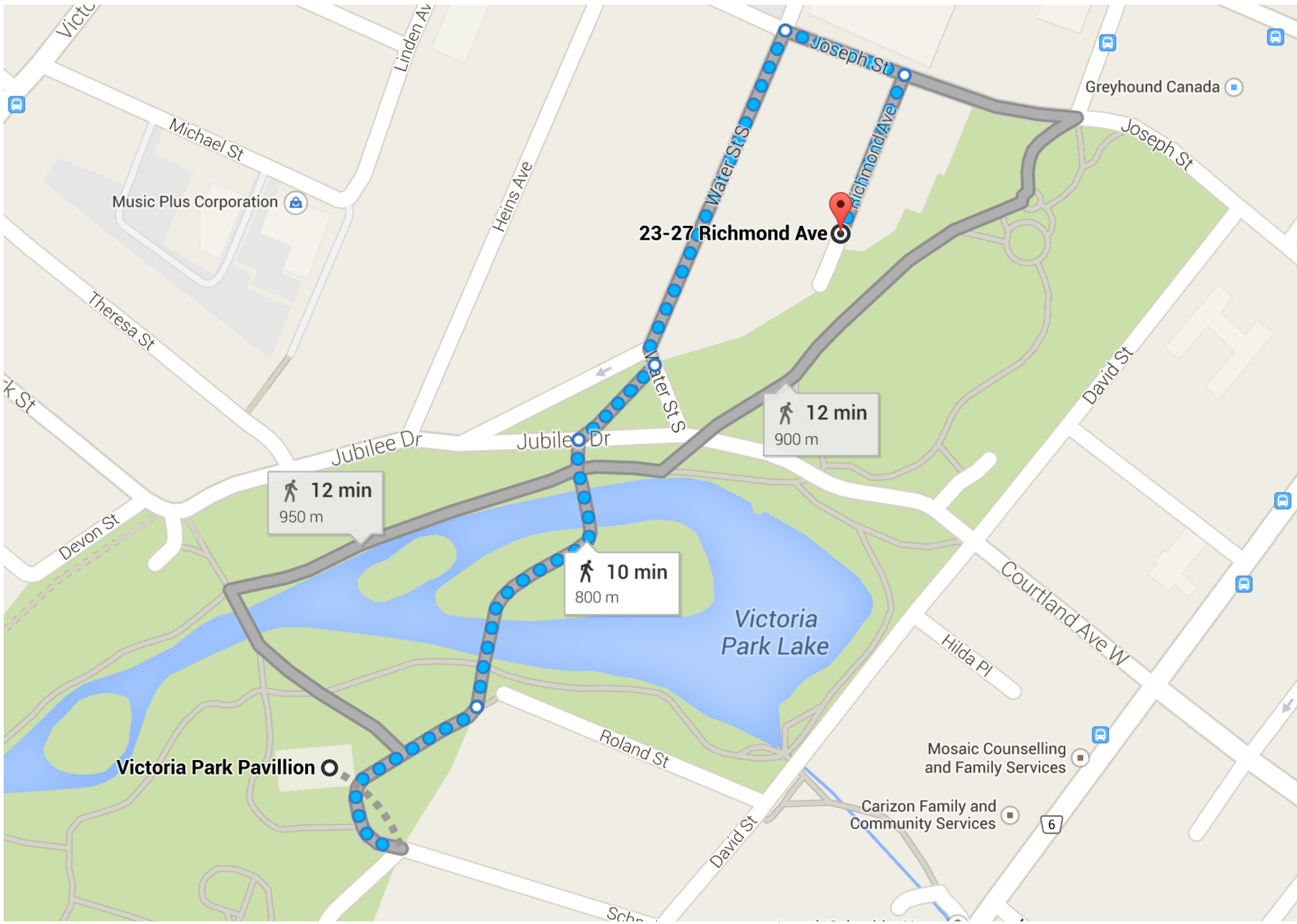
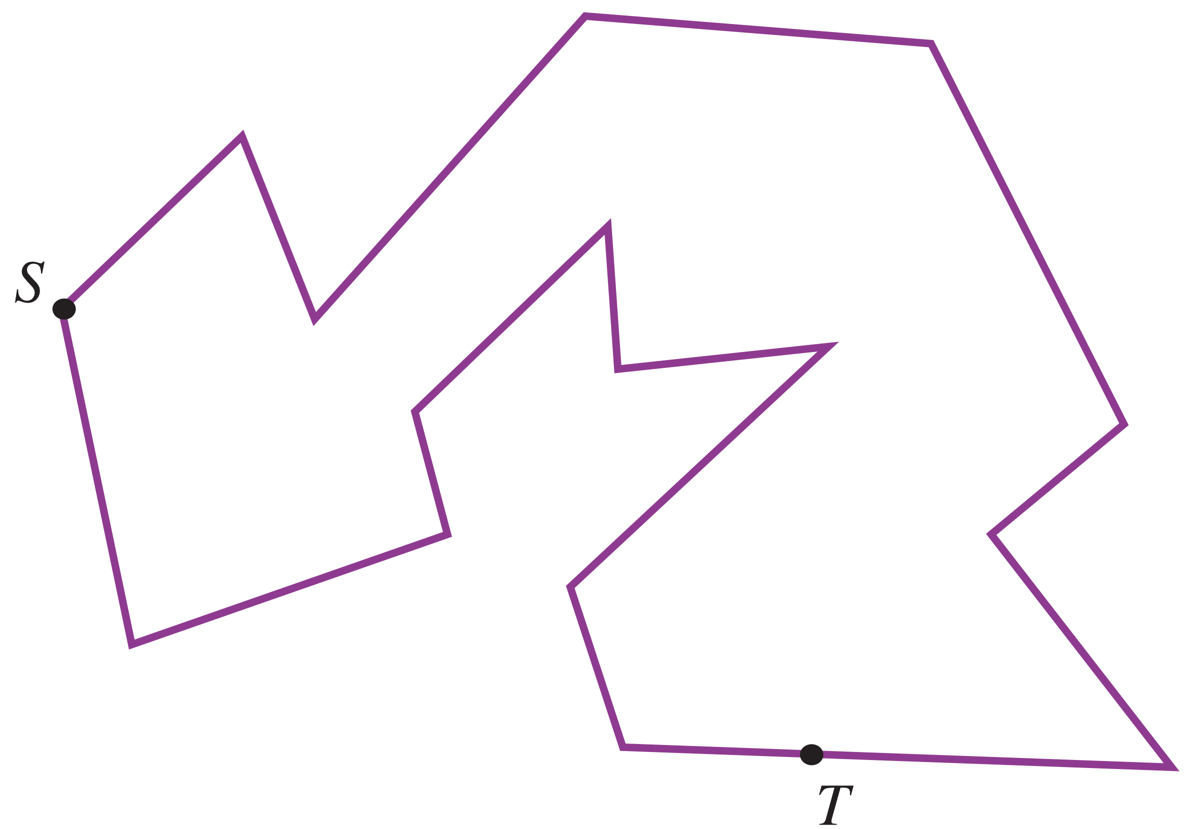
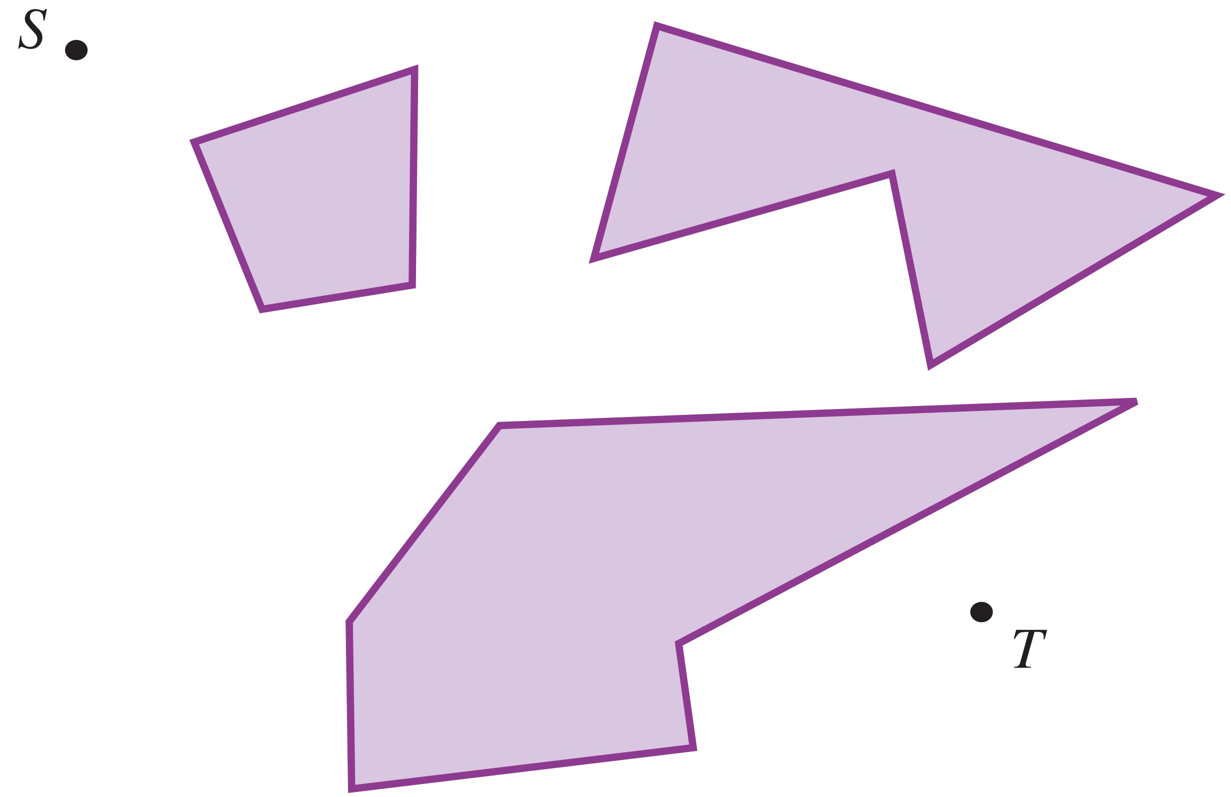


# Shortest Paths in 2D

## Polygon

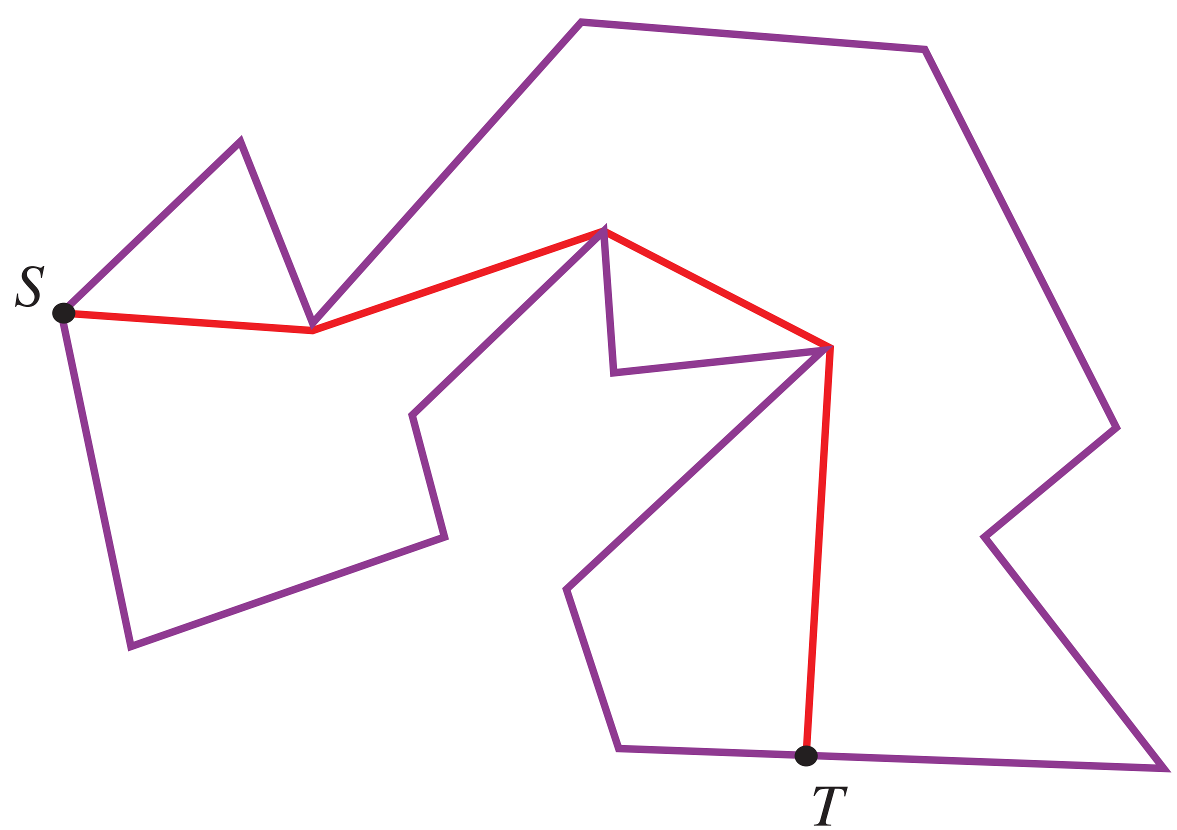



## Polygonal Domain



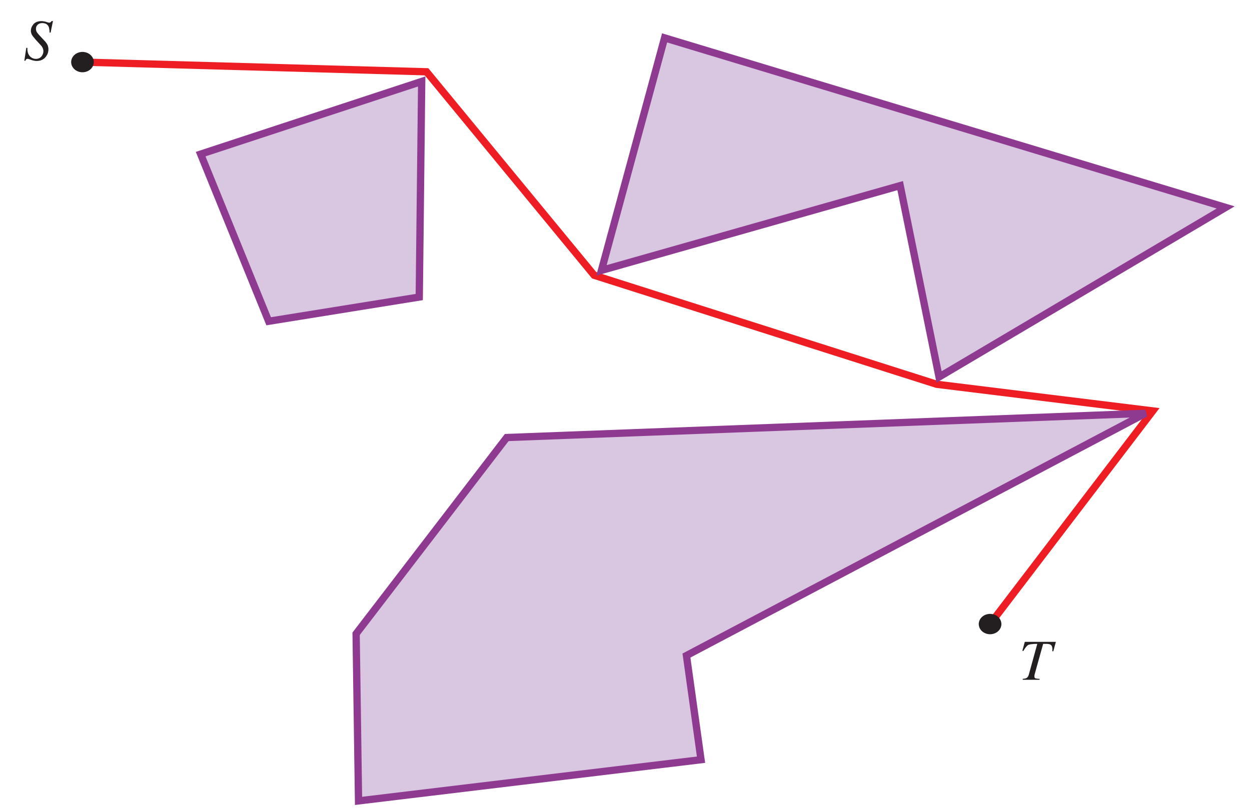
# Shortest Paths in 2D

## Polygon




$O(n)$

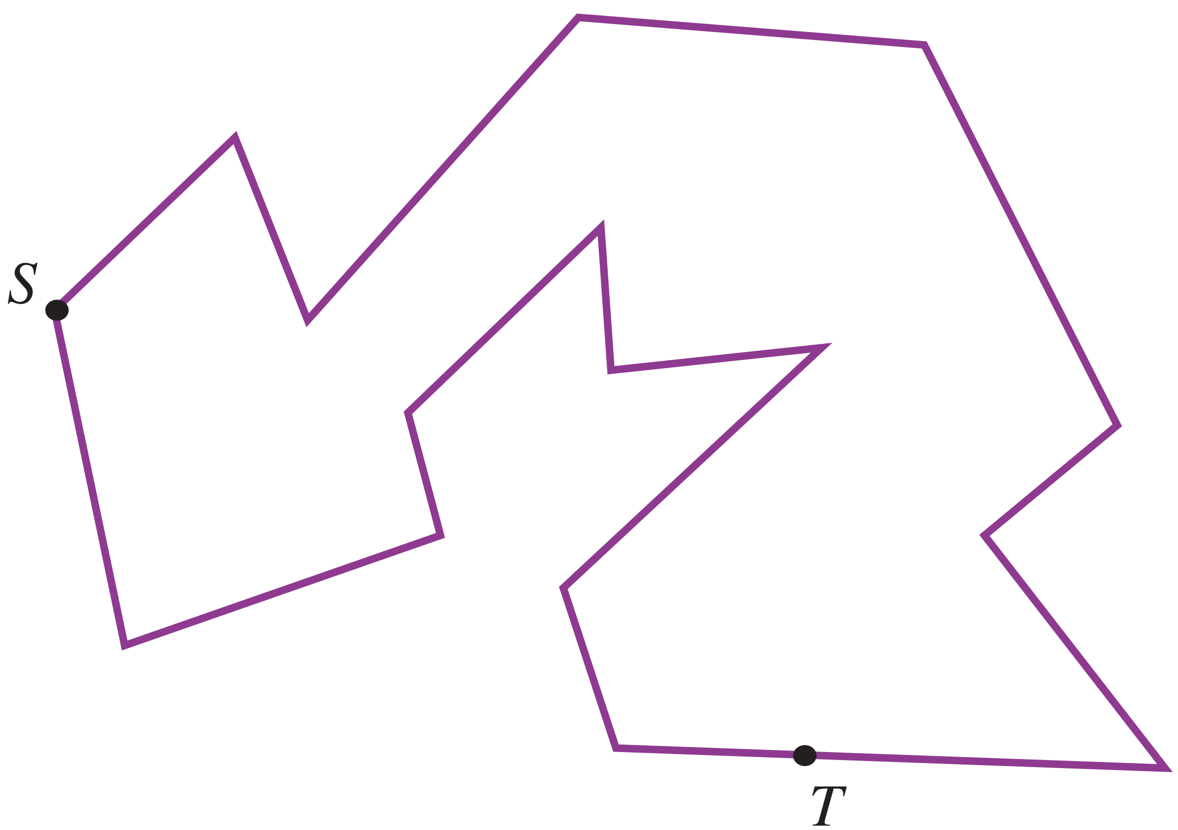
## Polygonal Domain




$O(n \log n)$

# Shortest Paths in a Polygon

Funnel Algorithm, $O(n)$

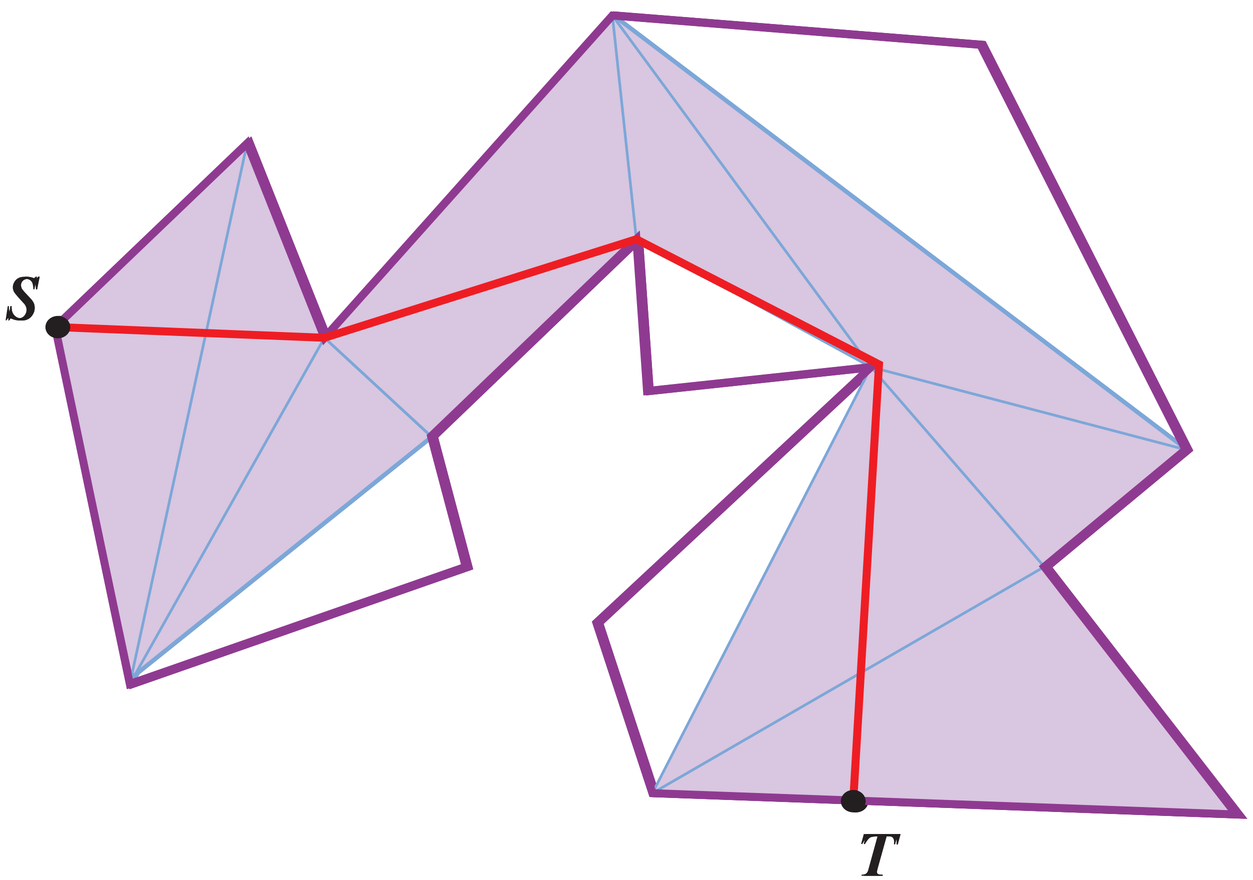



Idea:

- triangulate the polygon
- find path in triangulation
- narrow it down

# Shortest Paths in a Polygon

Funnel Algorithm, $O(n)$




Idea:

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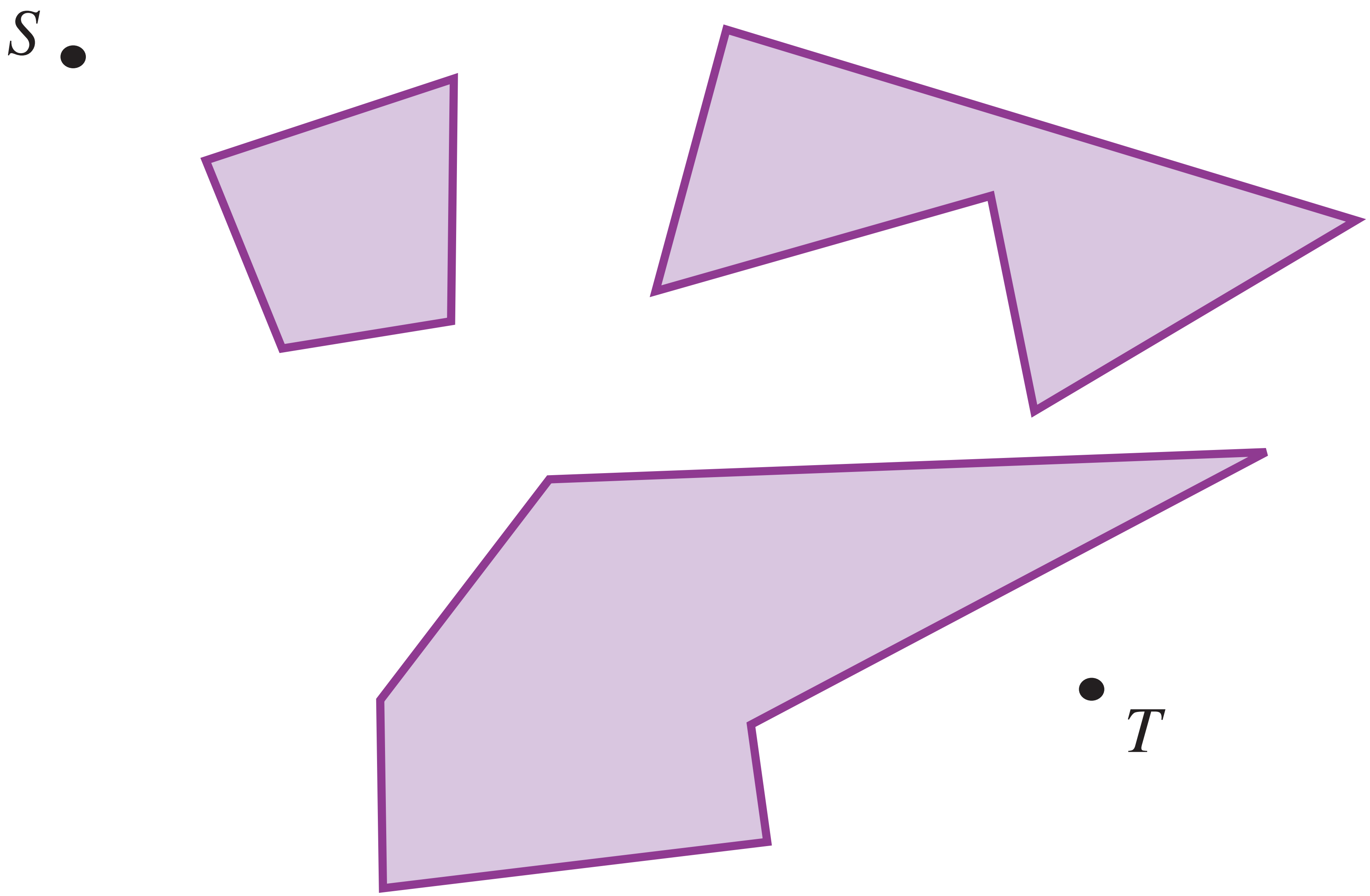
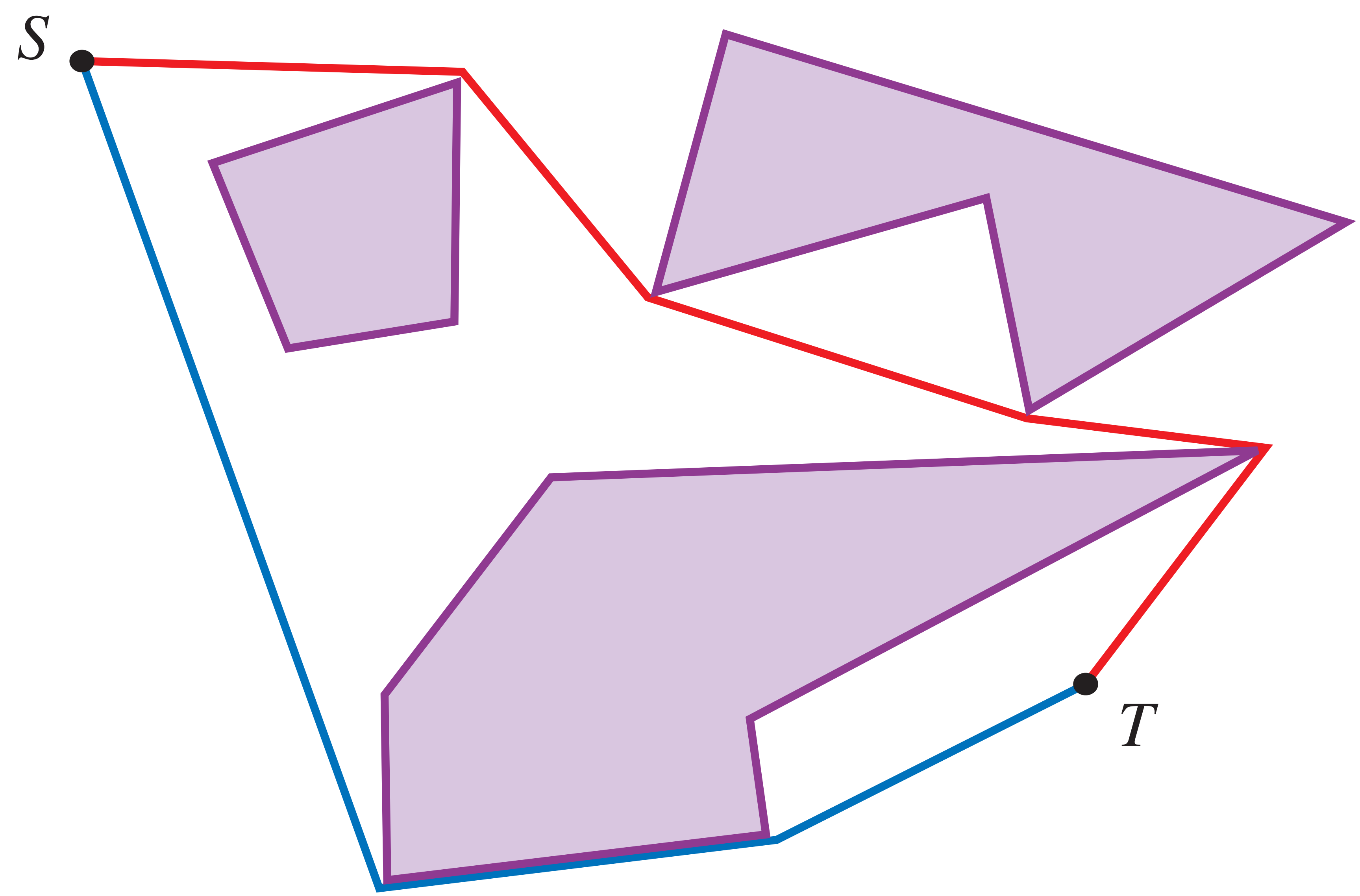
# Shortest Paths in the Plane with Obstacles



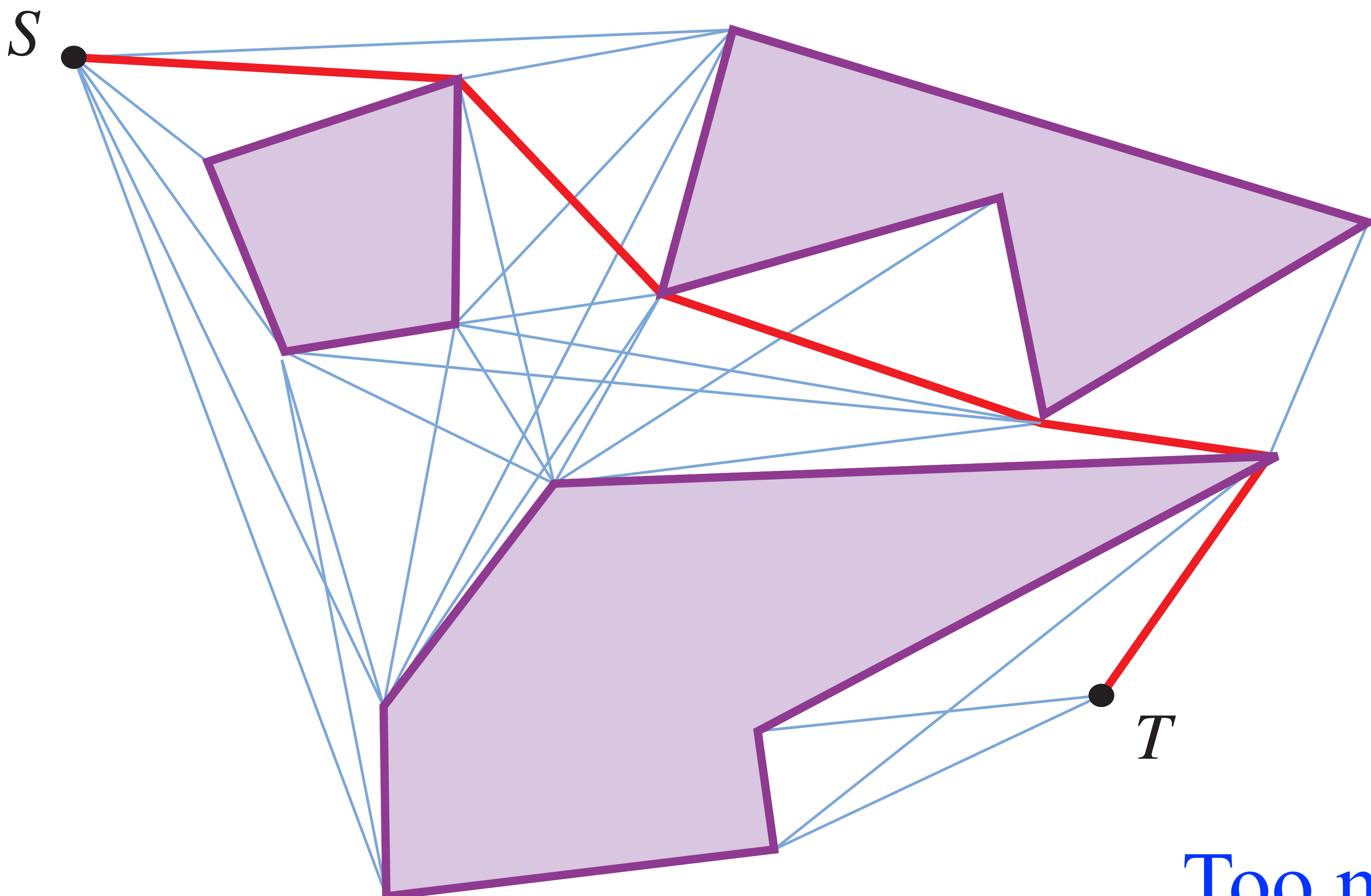
# Shortest Paths in the Plane with Obstacles




two locally shortest (“geodesic”) paths

# Shortest Paths in the Plane with Obstacles

## Using Dijkstra's graph algorithm.




Too many edges!

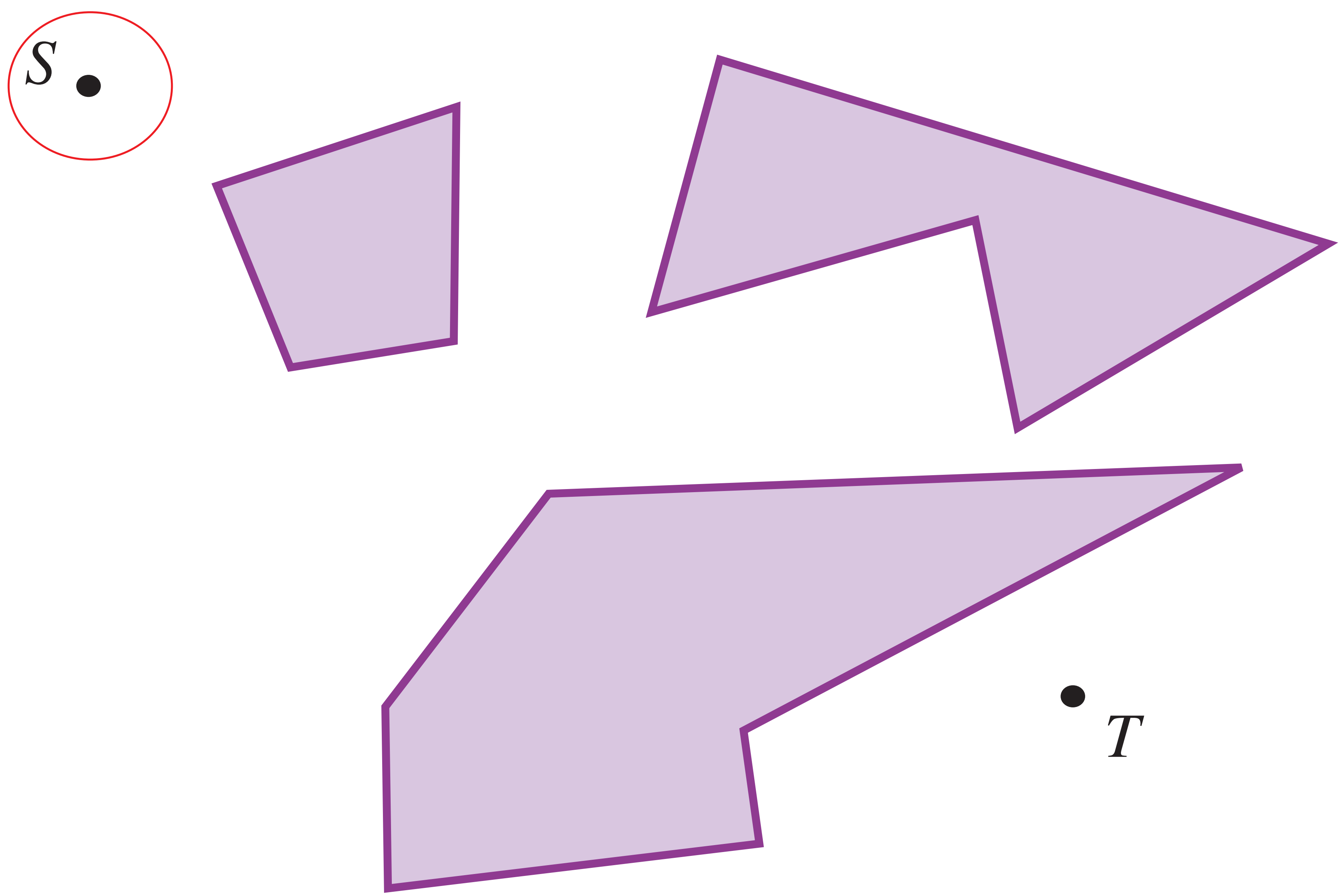
$O(n^2)$

Make a graph out of visible corners.

Apply Dijkstra's graph algorithm.

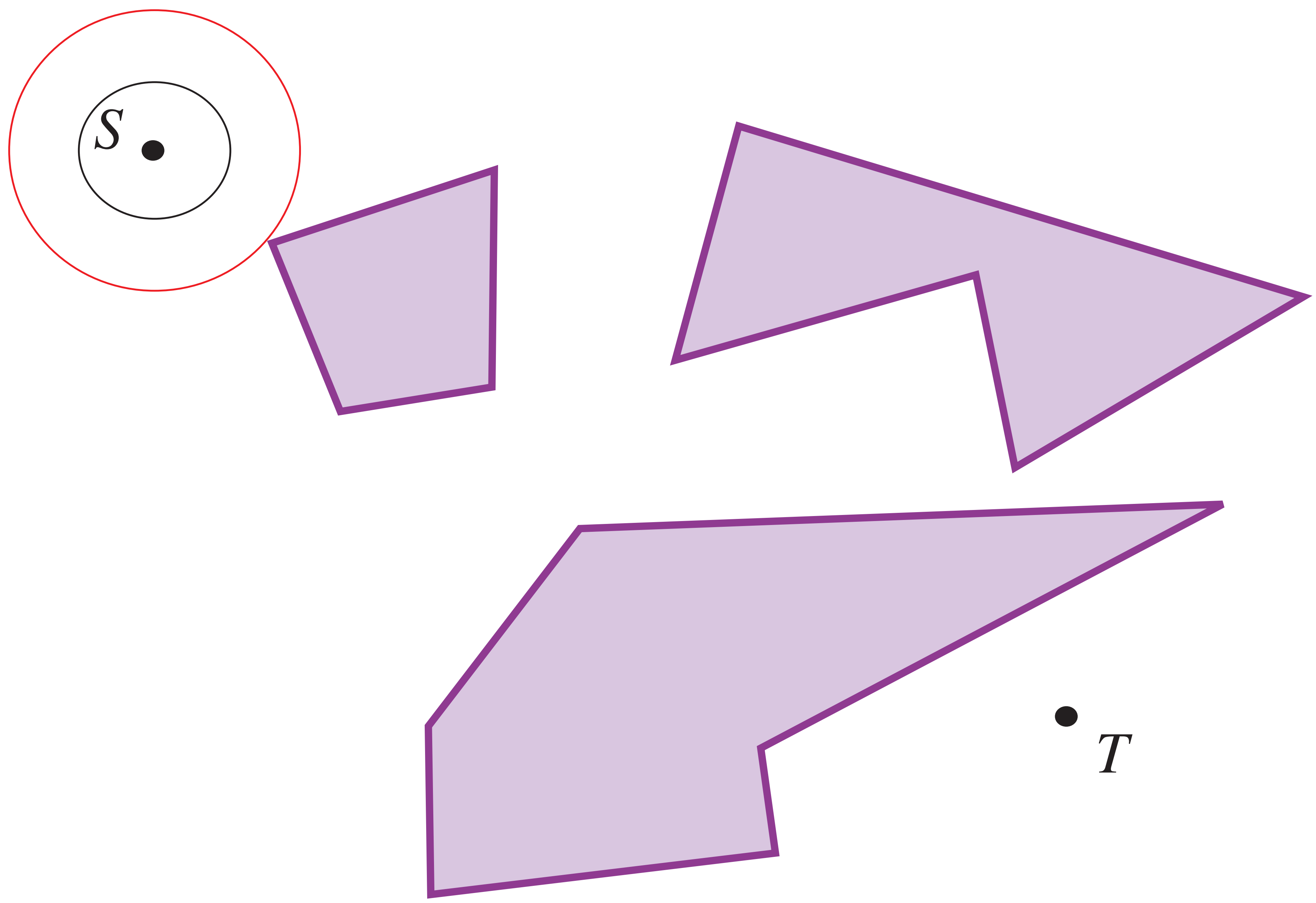
# Shortest Paths in the Plane

## Continuous Dijkstra



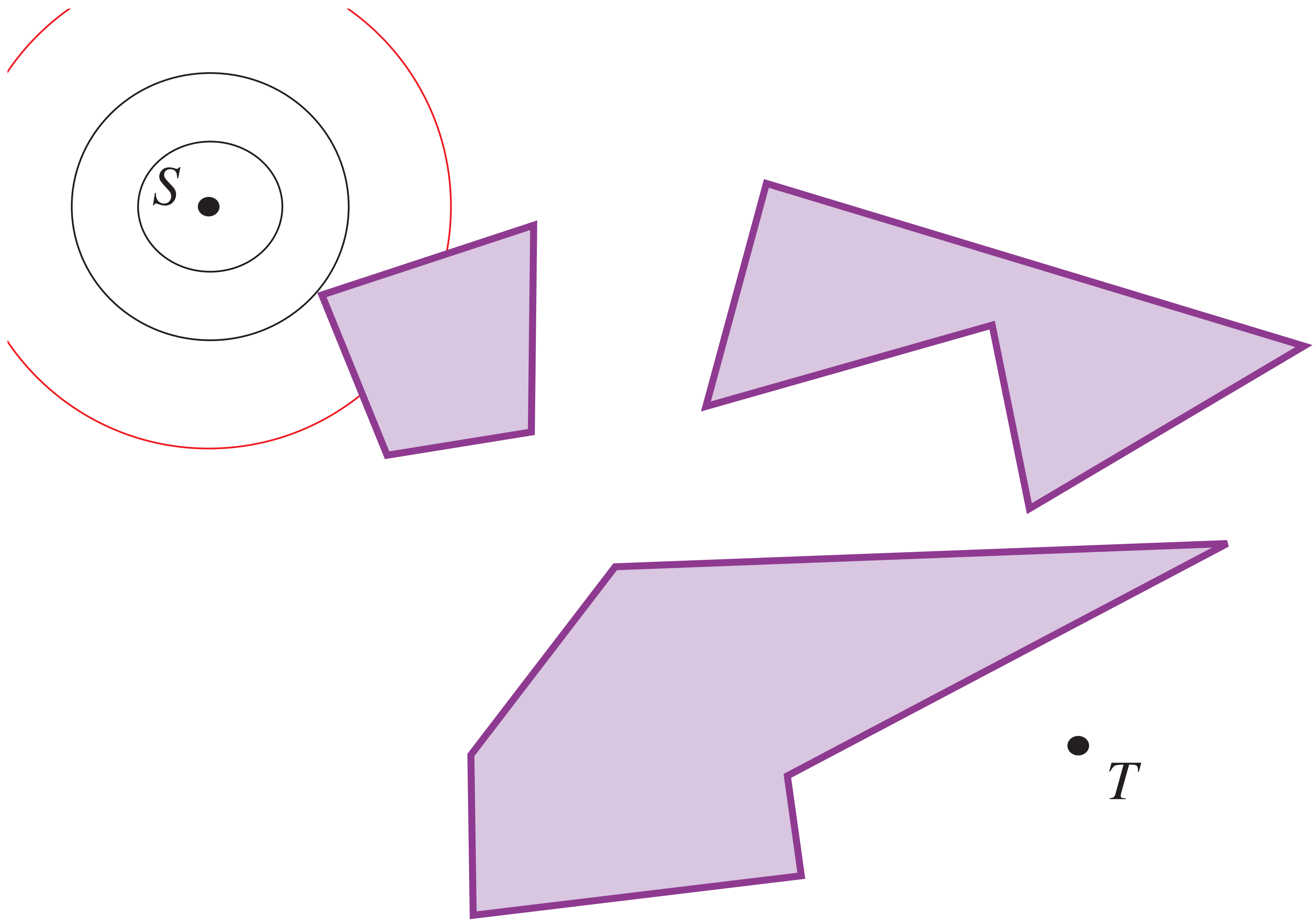
# Shortest Paths in the Plane

## Continuous Dijkstra



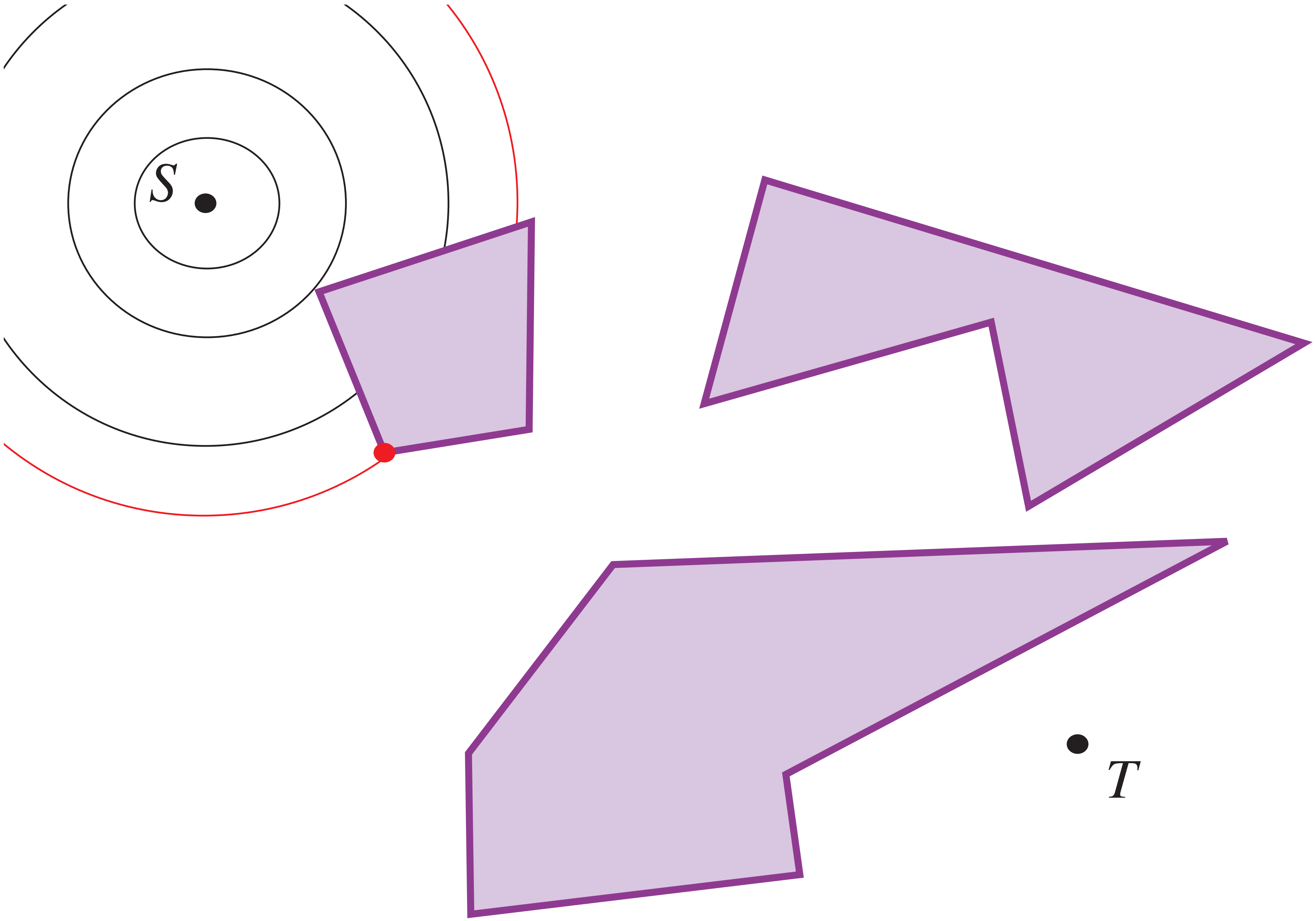
# Shortest Paths in the Plane

## Continuous Dijkstra



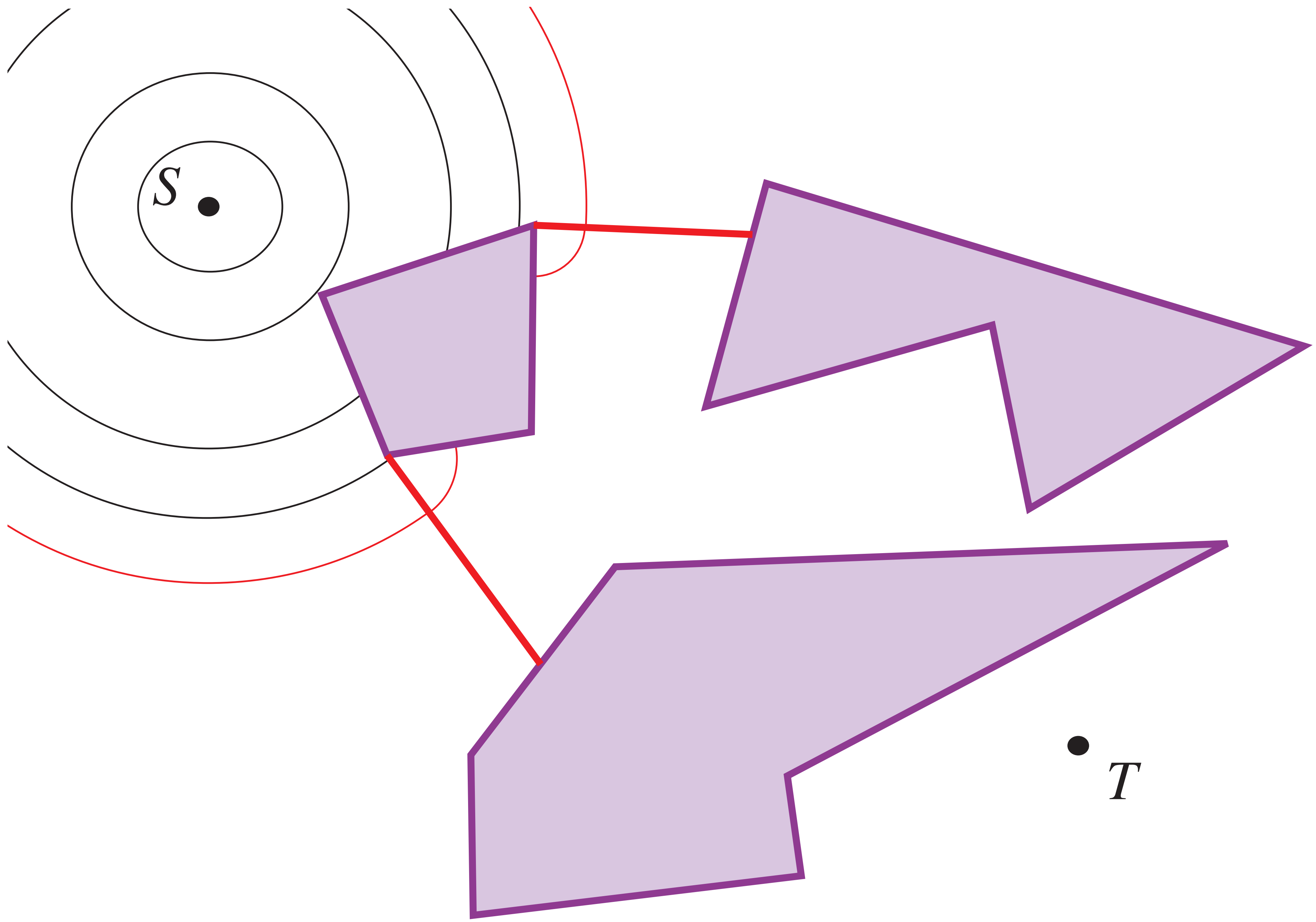
# Shortest Paths in the Plane

## Continuous Dijkstra



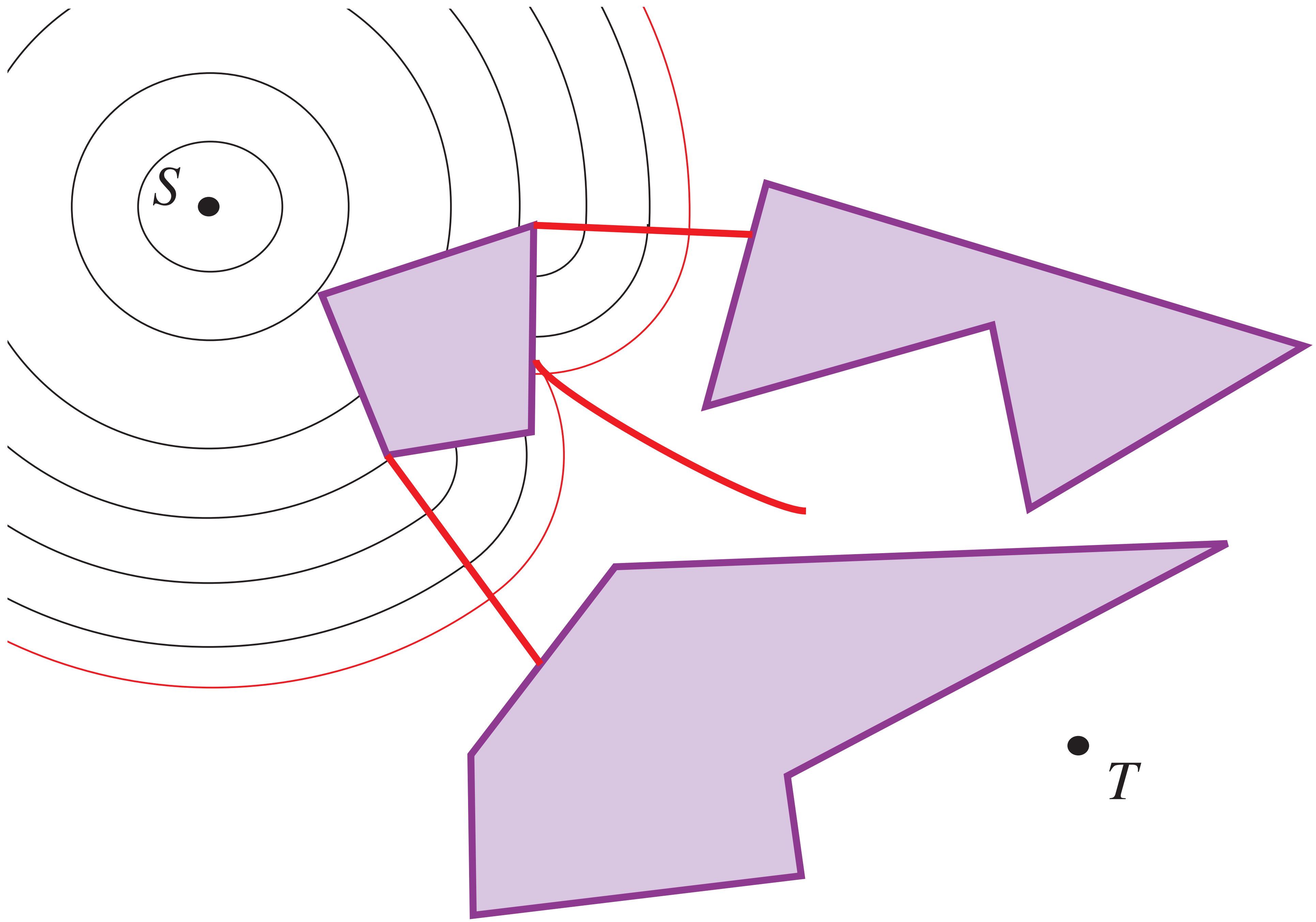
# Shortest Paths in the Plane

## Continuous Dijkstra



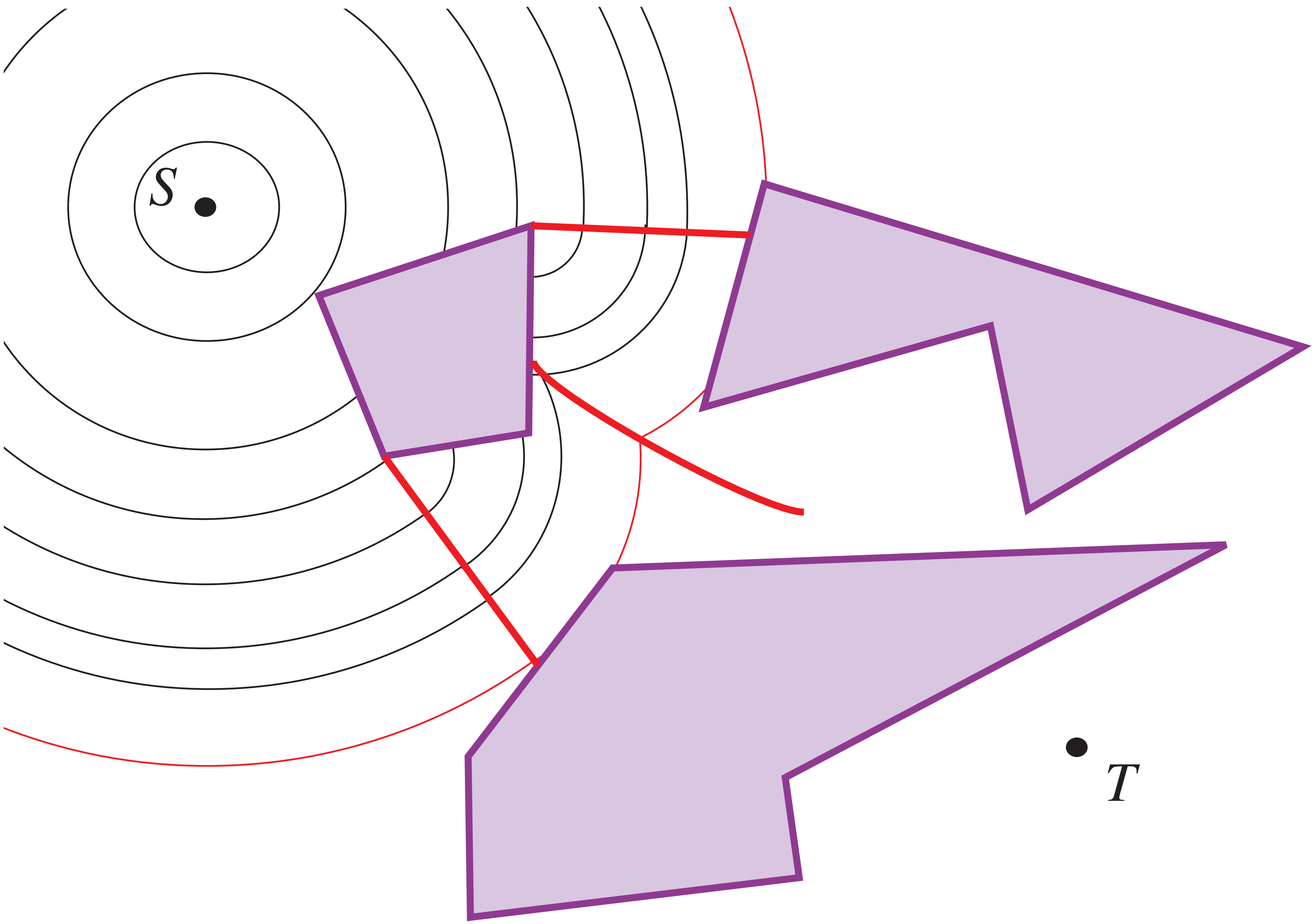
# Shortest Paths in the Plane

## Continuous Dijkstra



# Shortest Paths in the Plane

## Continuous Dijkstra



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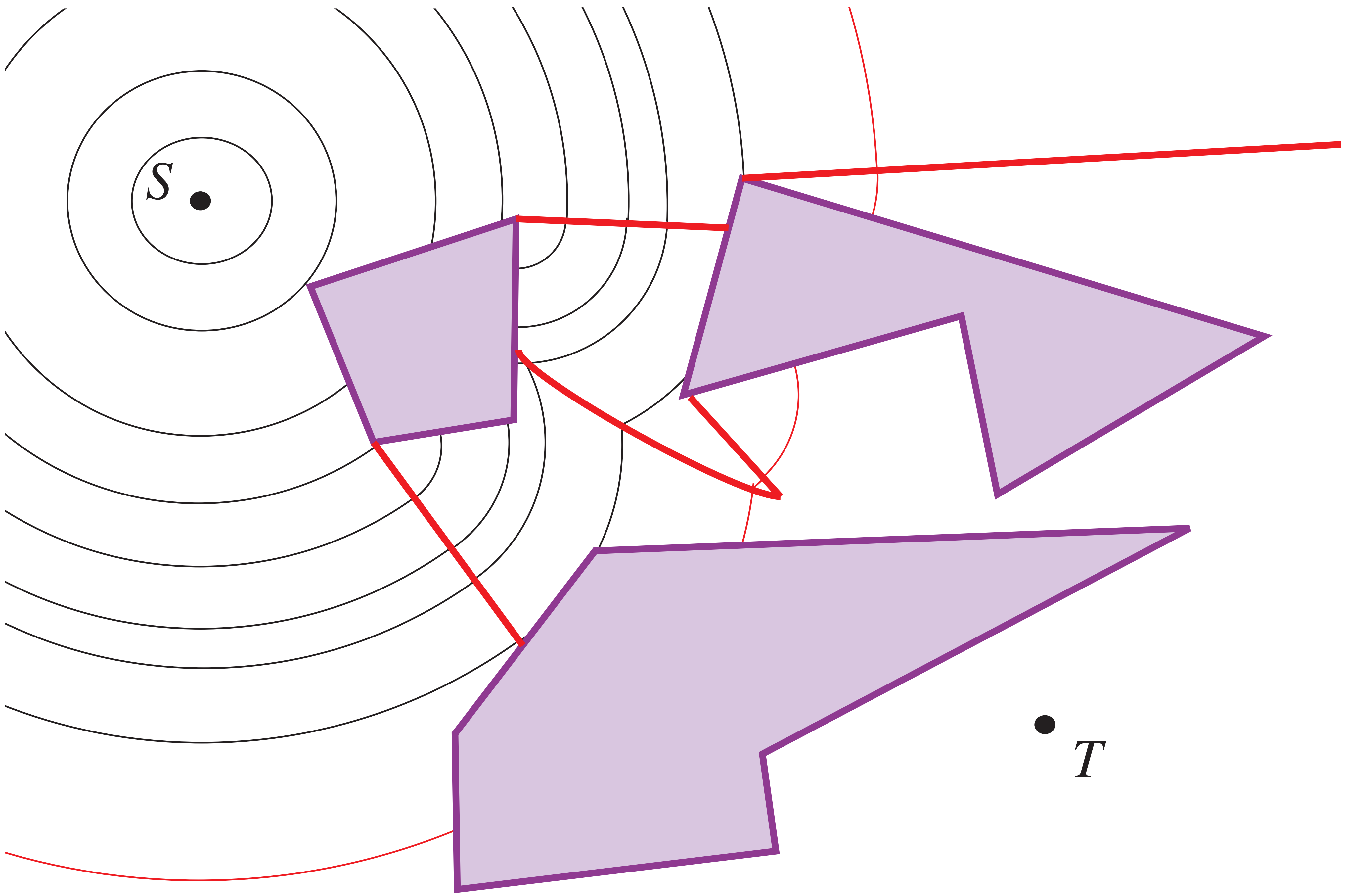
## Continuous Dijkstra



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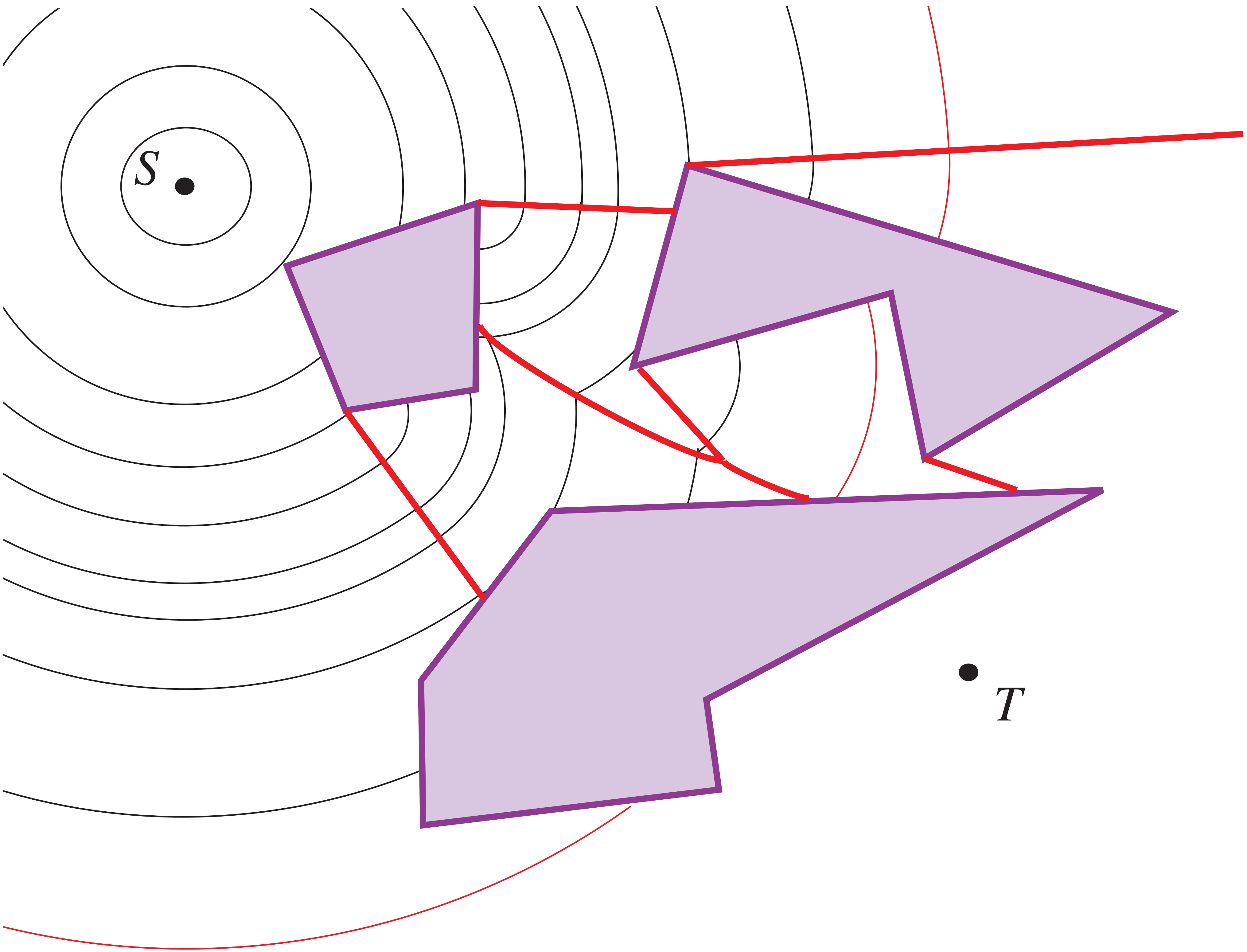
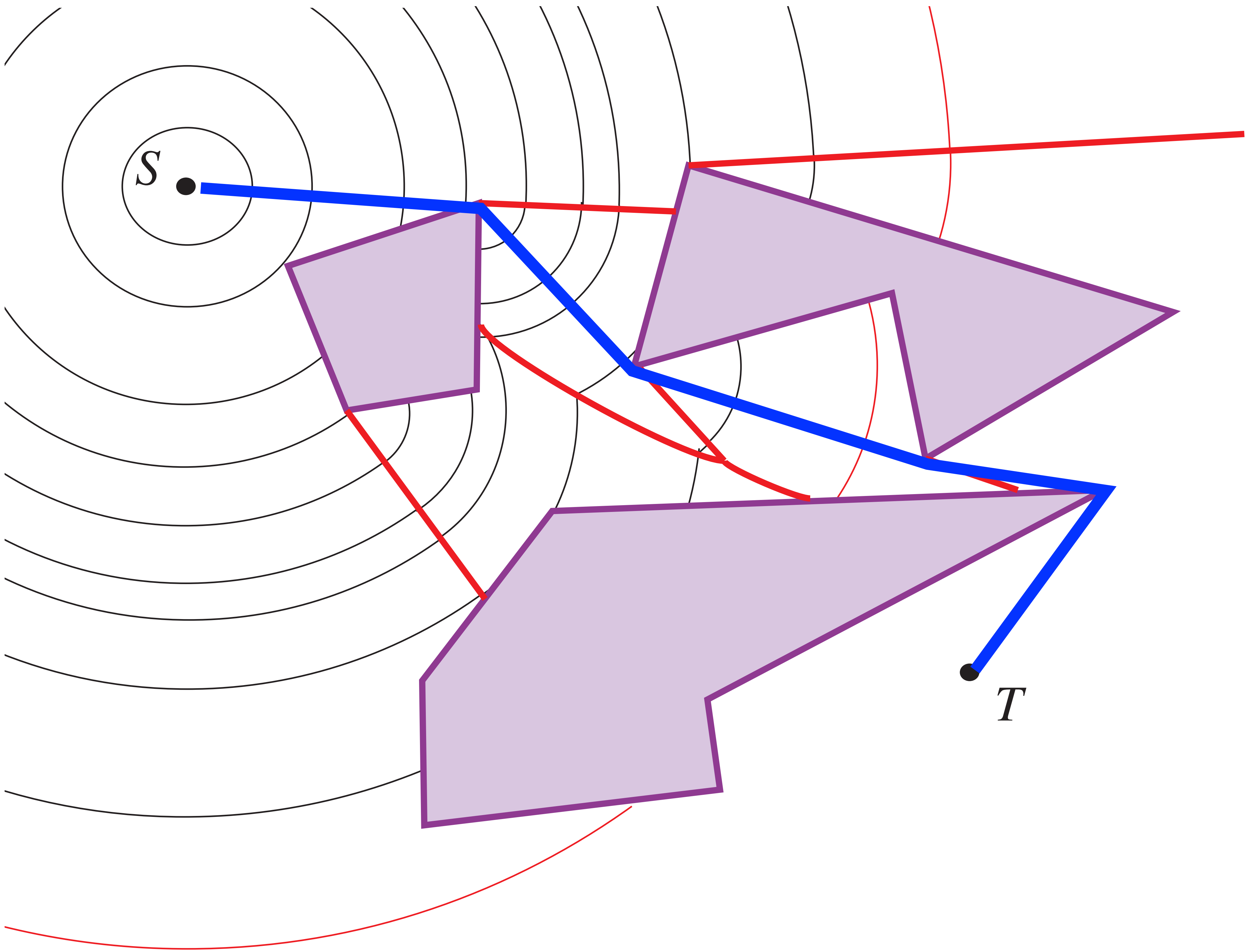
## Continuous Dijkstra

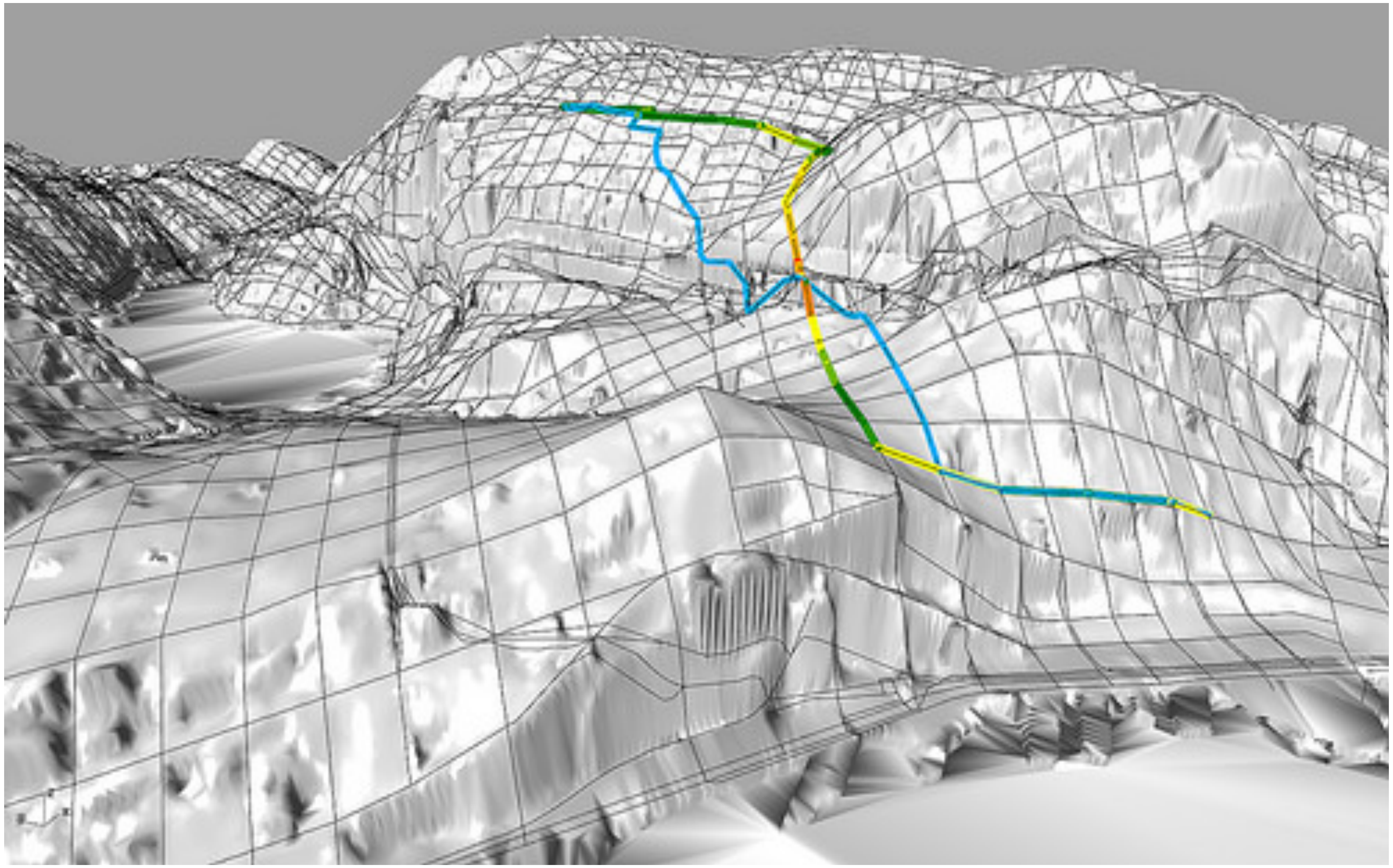


# Shortest Paths in the Plane

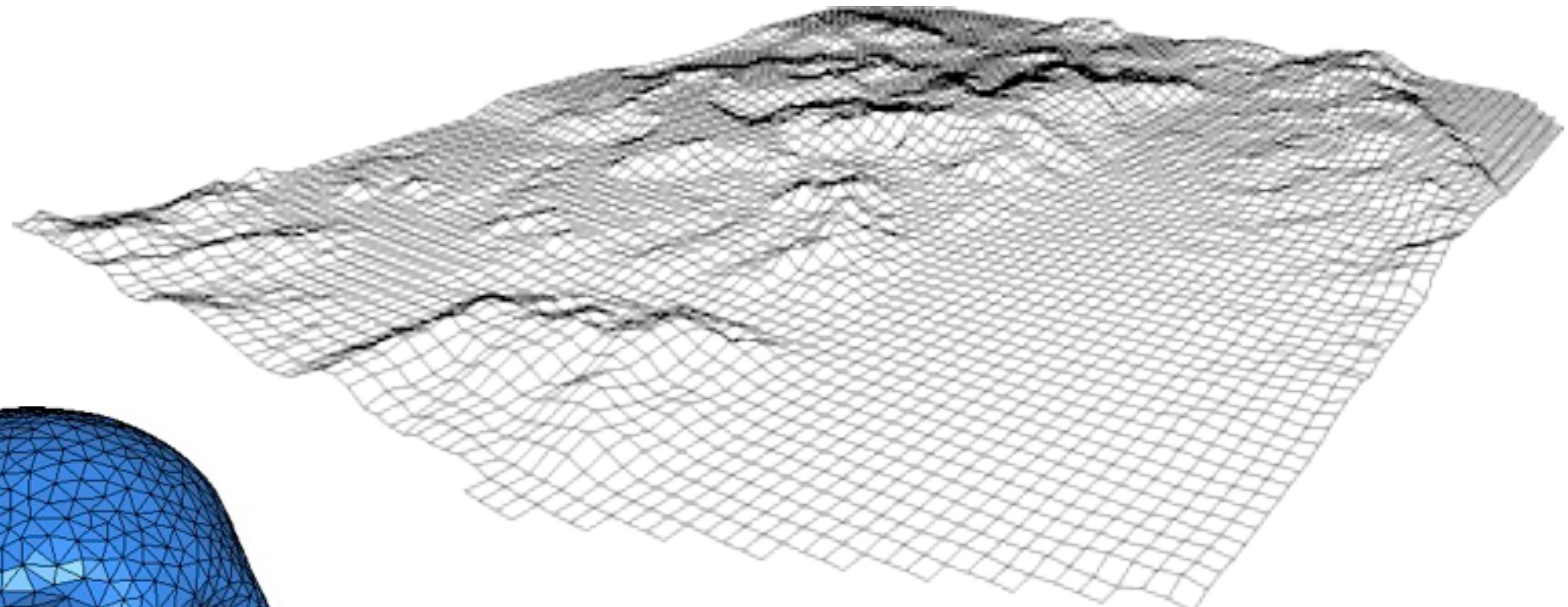
Continuous Dijkstra  *O*(*n* log *n*)  Mitchell, Hershberger & Suri, ‘93



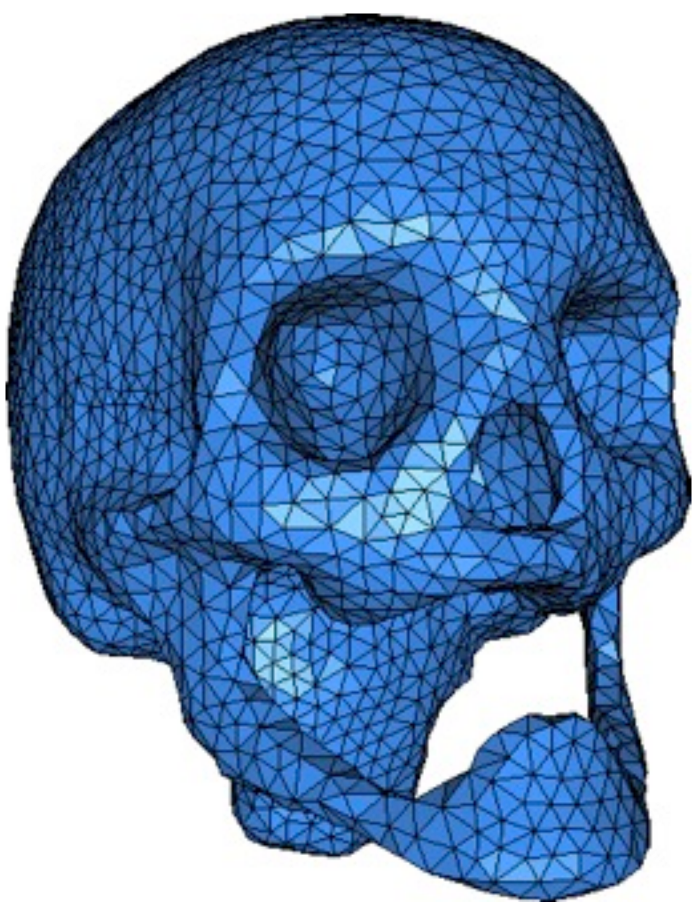
# Moving Away from Flatland



# Shortest Paths in 3D – Polyhedral Surfaces



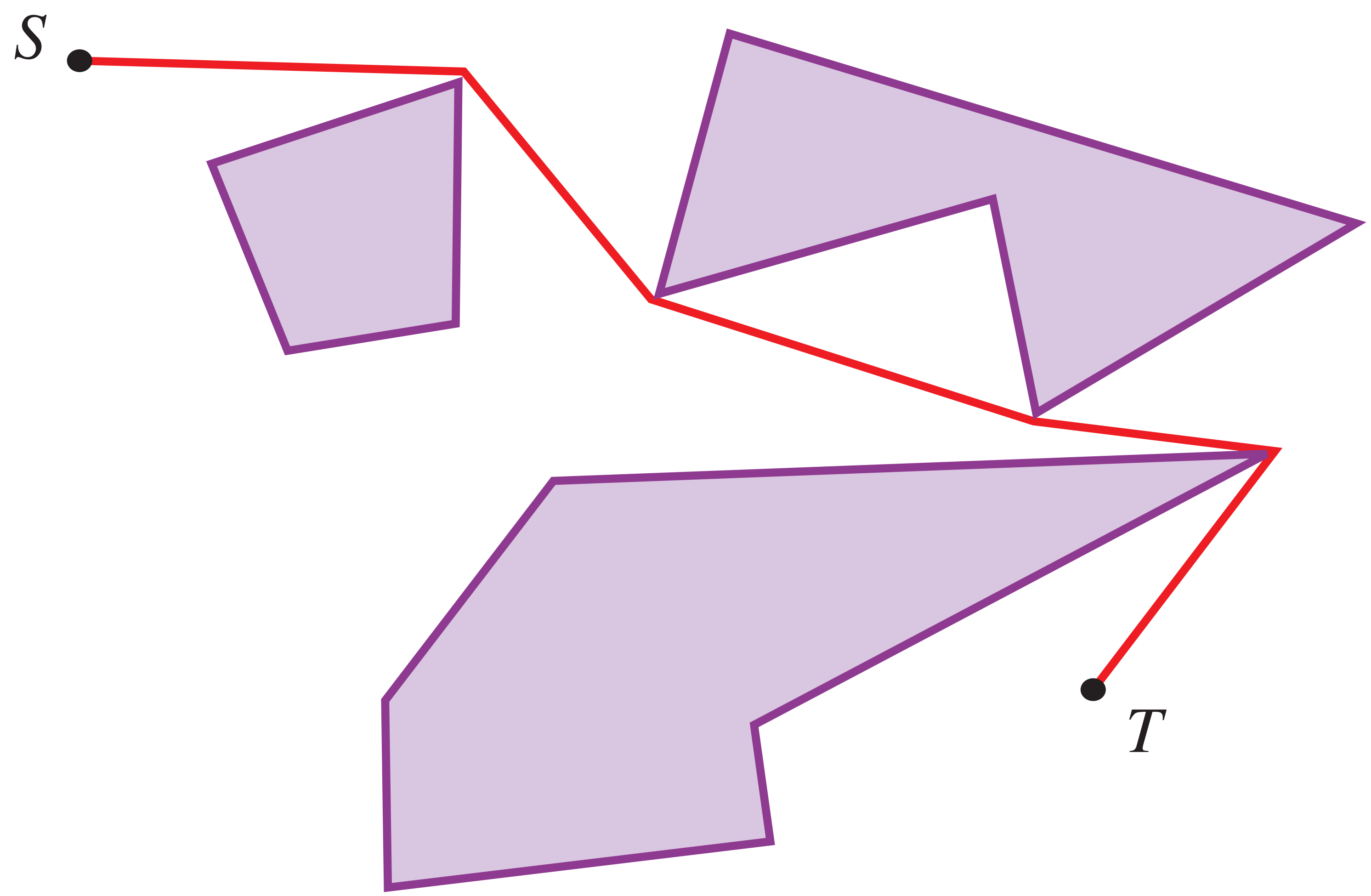
Paul Bourke



Rineau and Yvinec

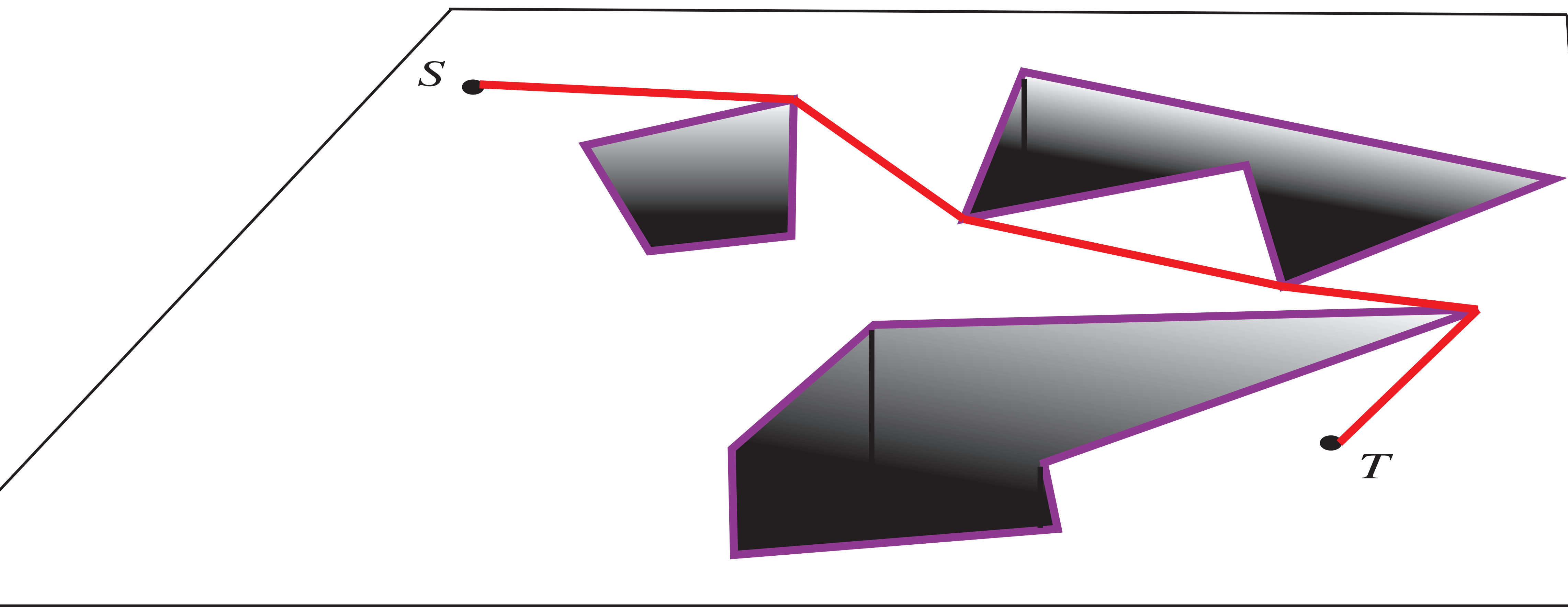
# Shortest Paths in 3D – Polyhedral Surfaces

why does this generalize 2D with obstacles?



# Shortest Paths in 3D – Polyhedral Surfaces

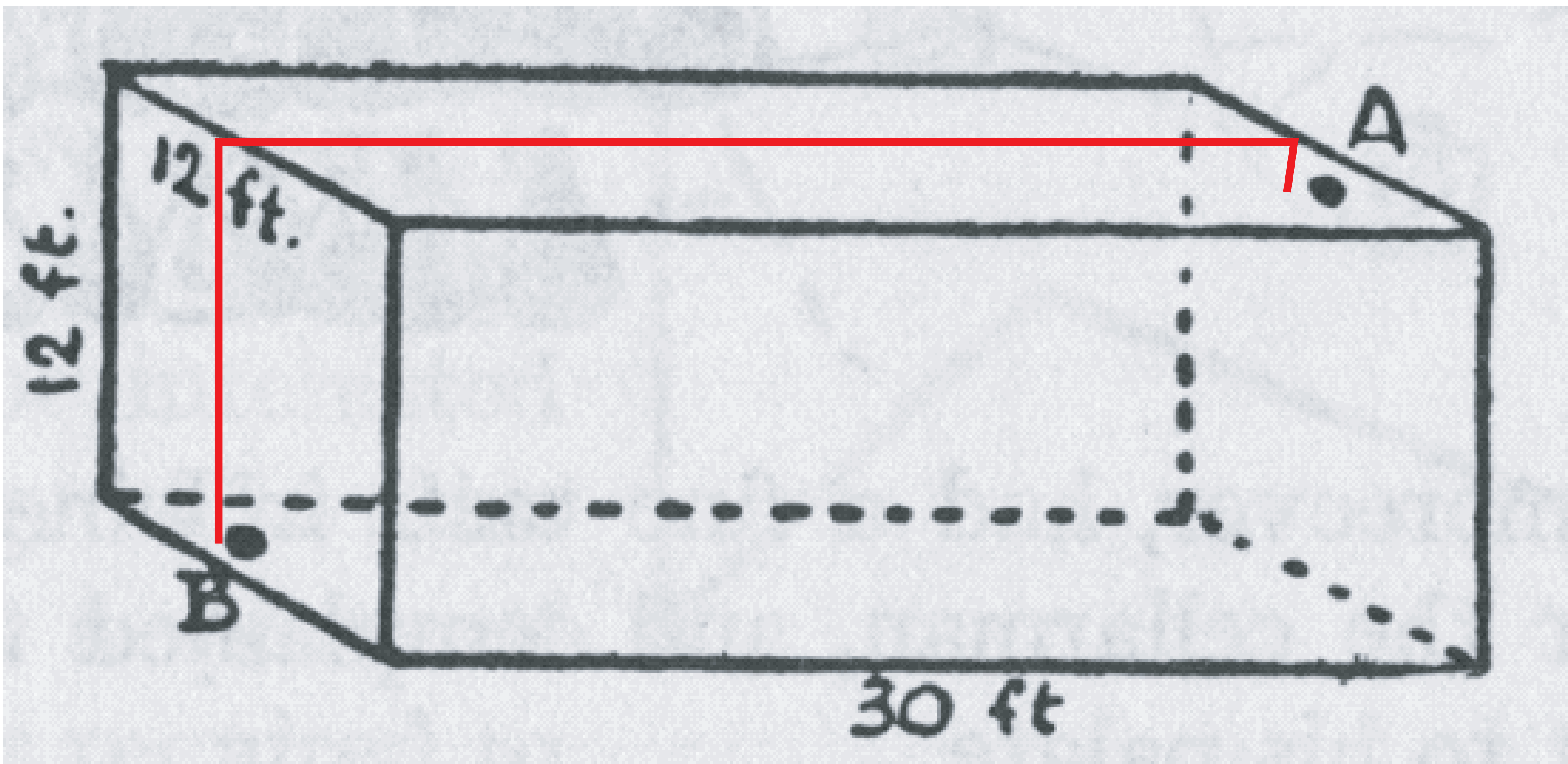
why does this generalize 2D with obstacles?



# The spider and the fly problem

Dudeney, The Canterbury Puzzles, 1958

?



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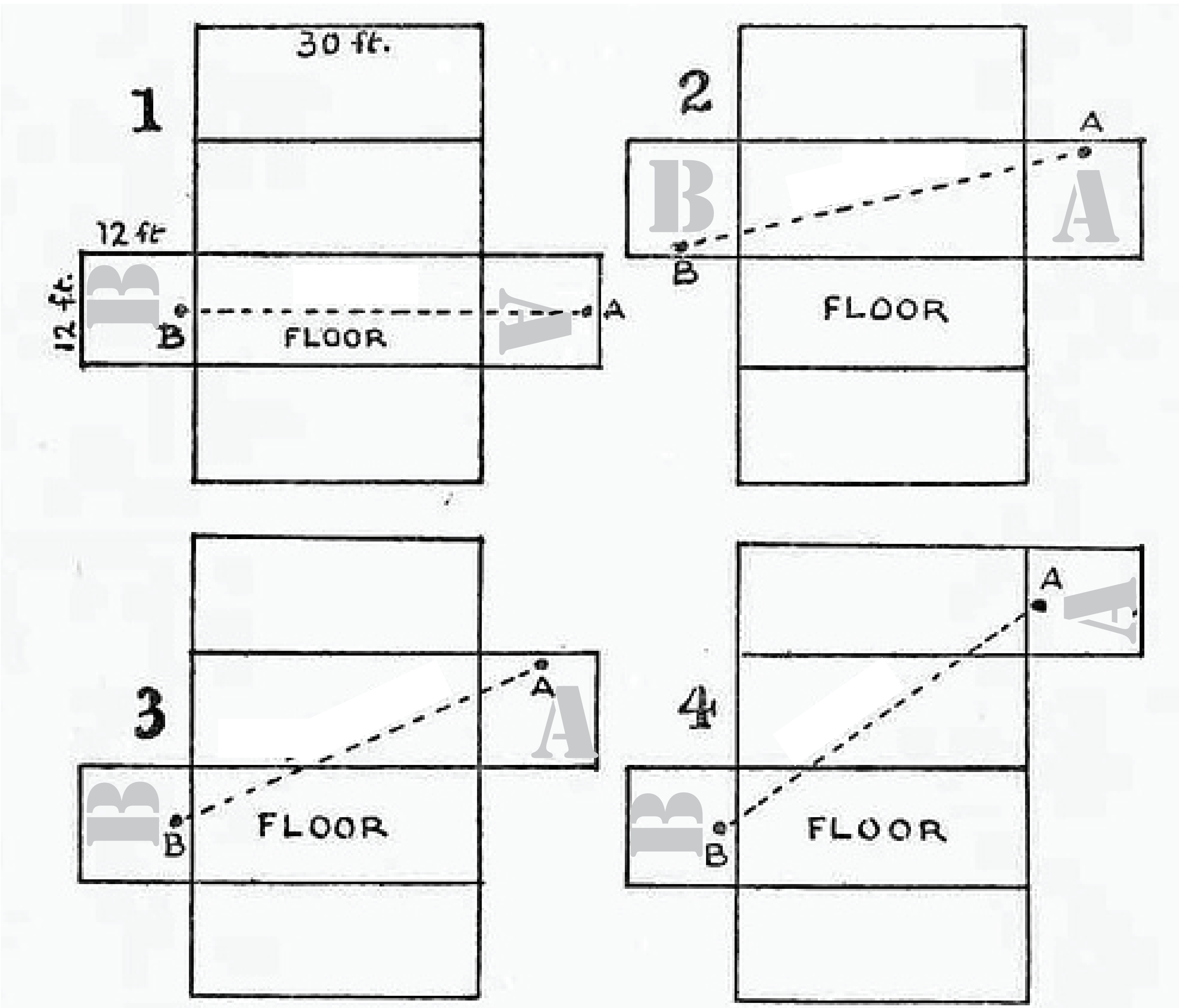
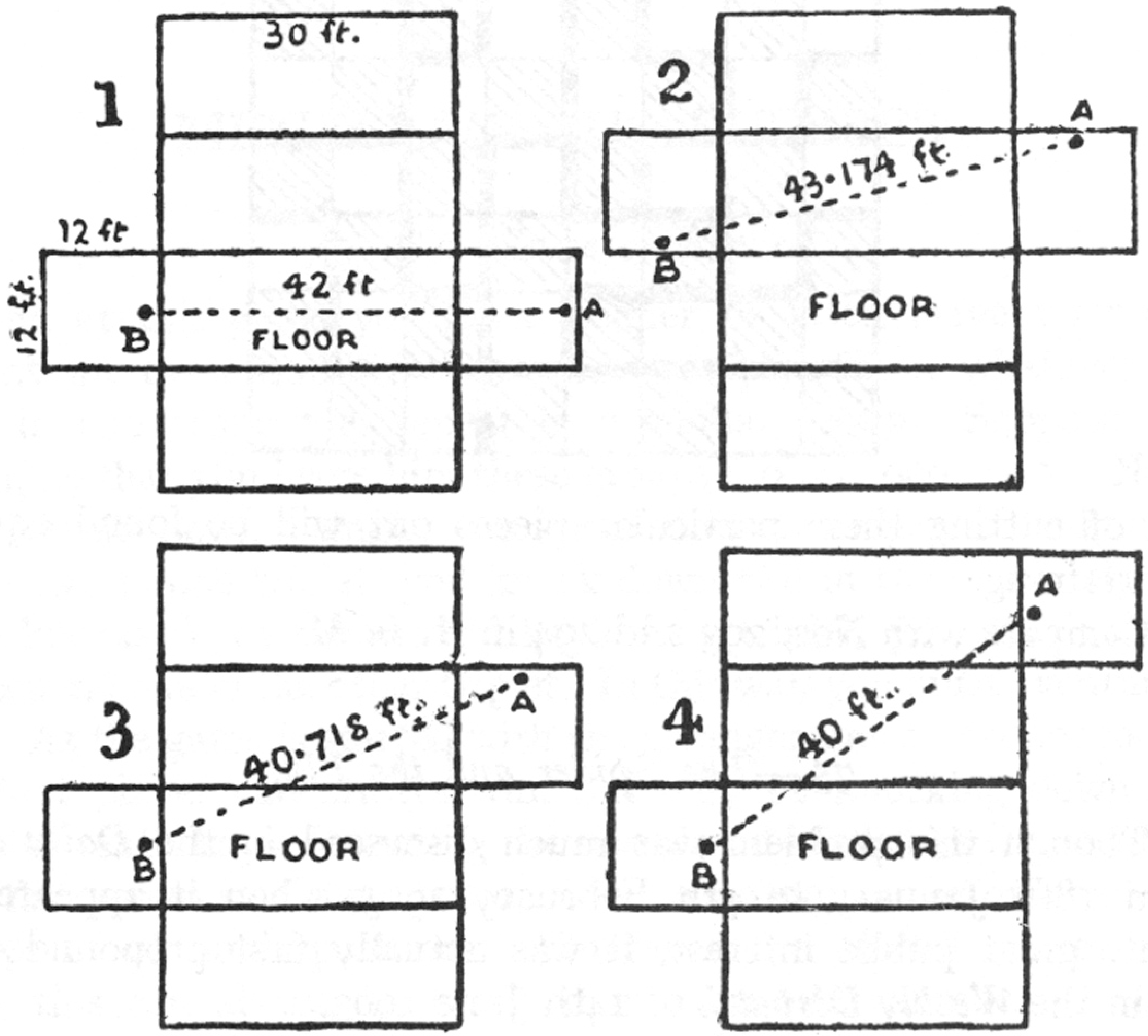


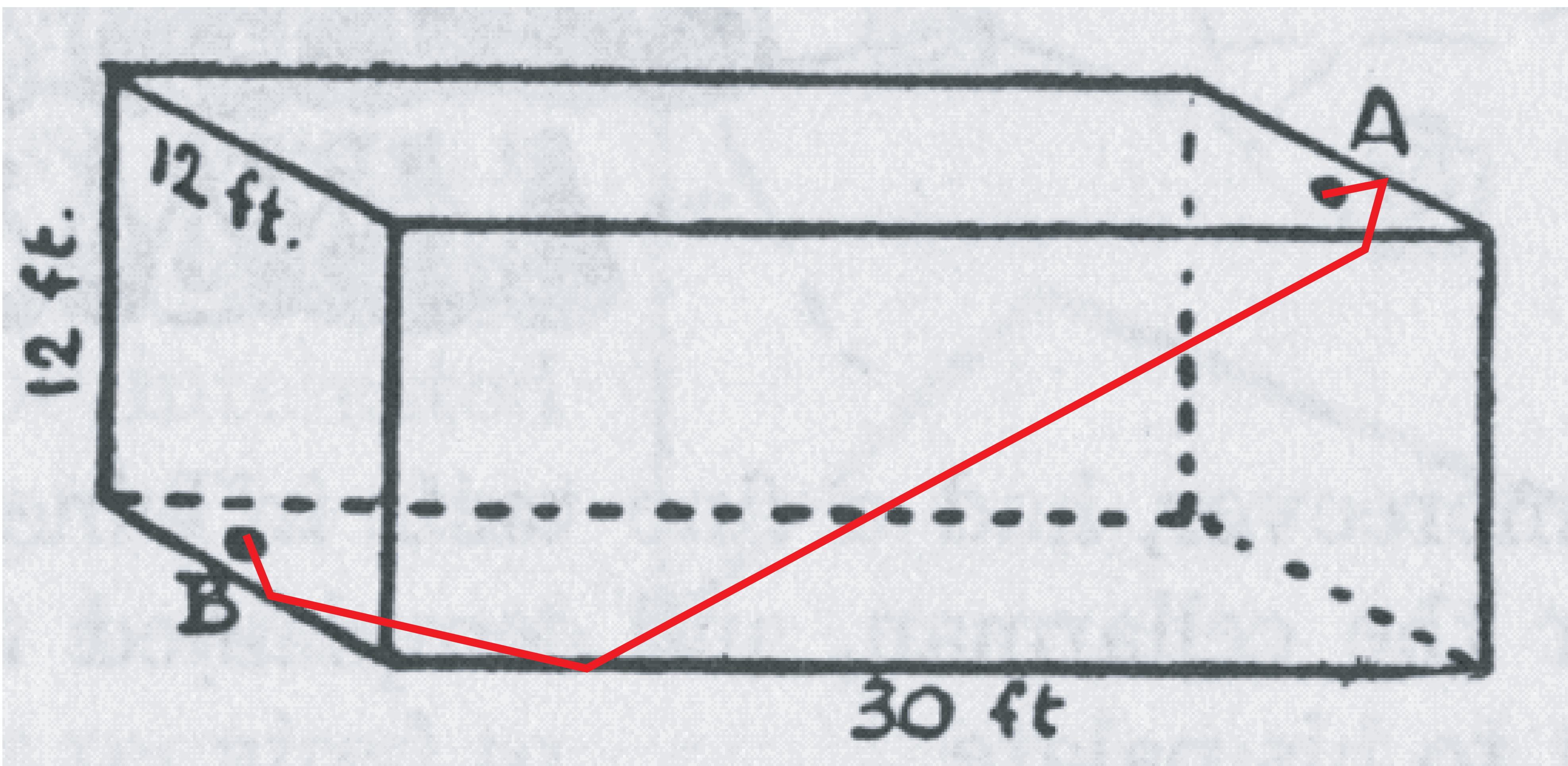

locally shortest paths are straight lines in unfoldings

# The spider and the fly problem



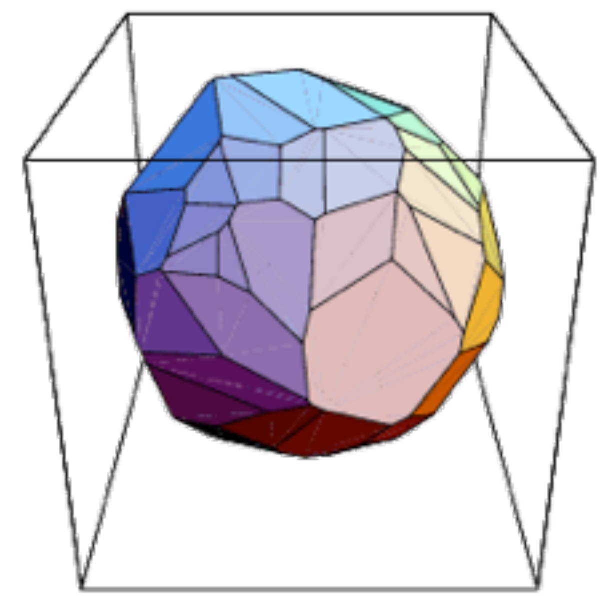
# Shortest Paths on Polyhedon

the spider and the fly problem



# Shortest Paths on Polyhedon

running times of algorithms

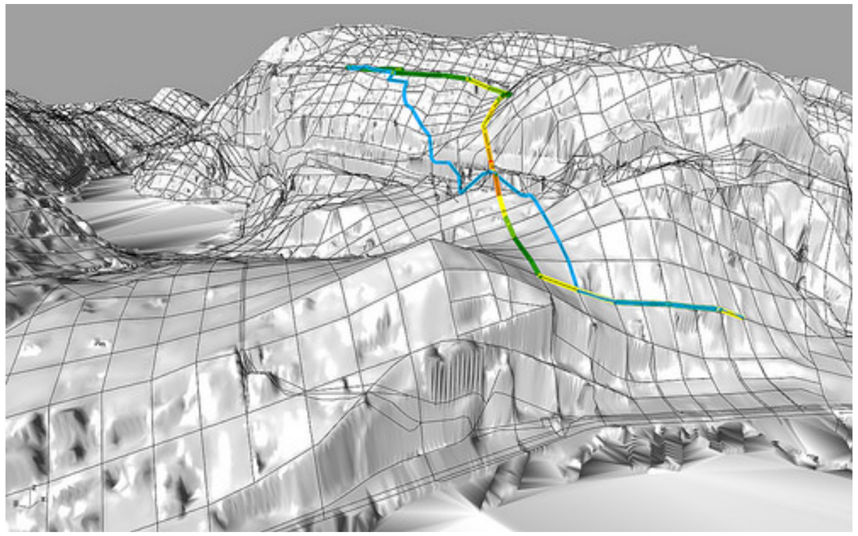


Shortest paths on a convex polyhedron

$O(n \log n)$ Schreiber and Sharir `08

Shortest paths on a general polyhedron/polyhedral terrain

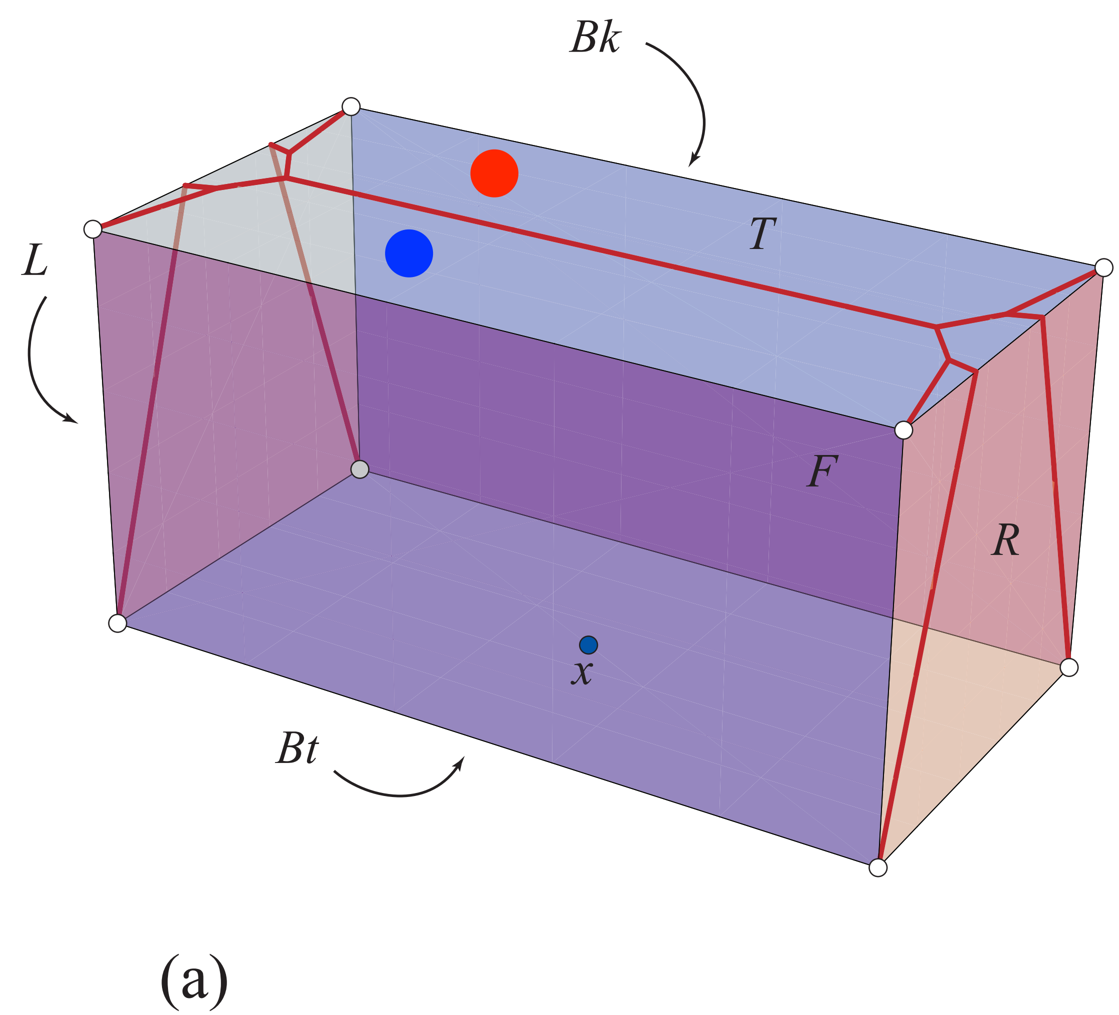
$O(n^2)$ Chen and Han `96

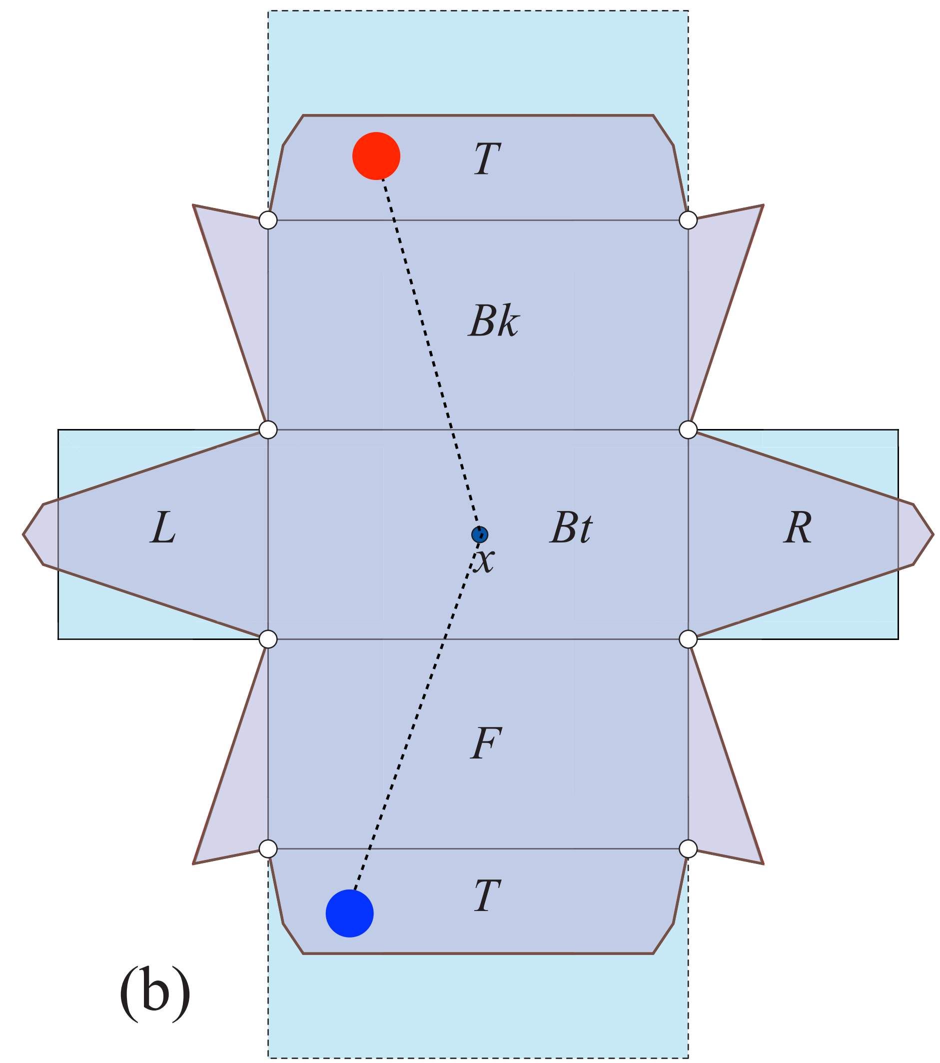


# Shortest Paths on Polyhedral Surfaces

Fast Exact and Approximate Geodesics on Meshes
SIGRAPH 2005

# Shortest Paths on Convex Polyhedron

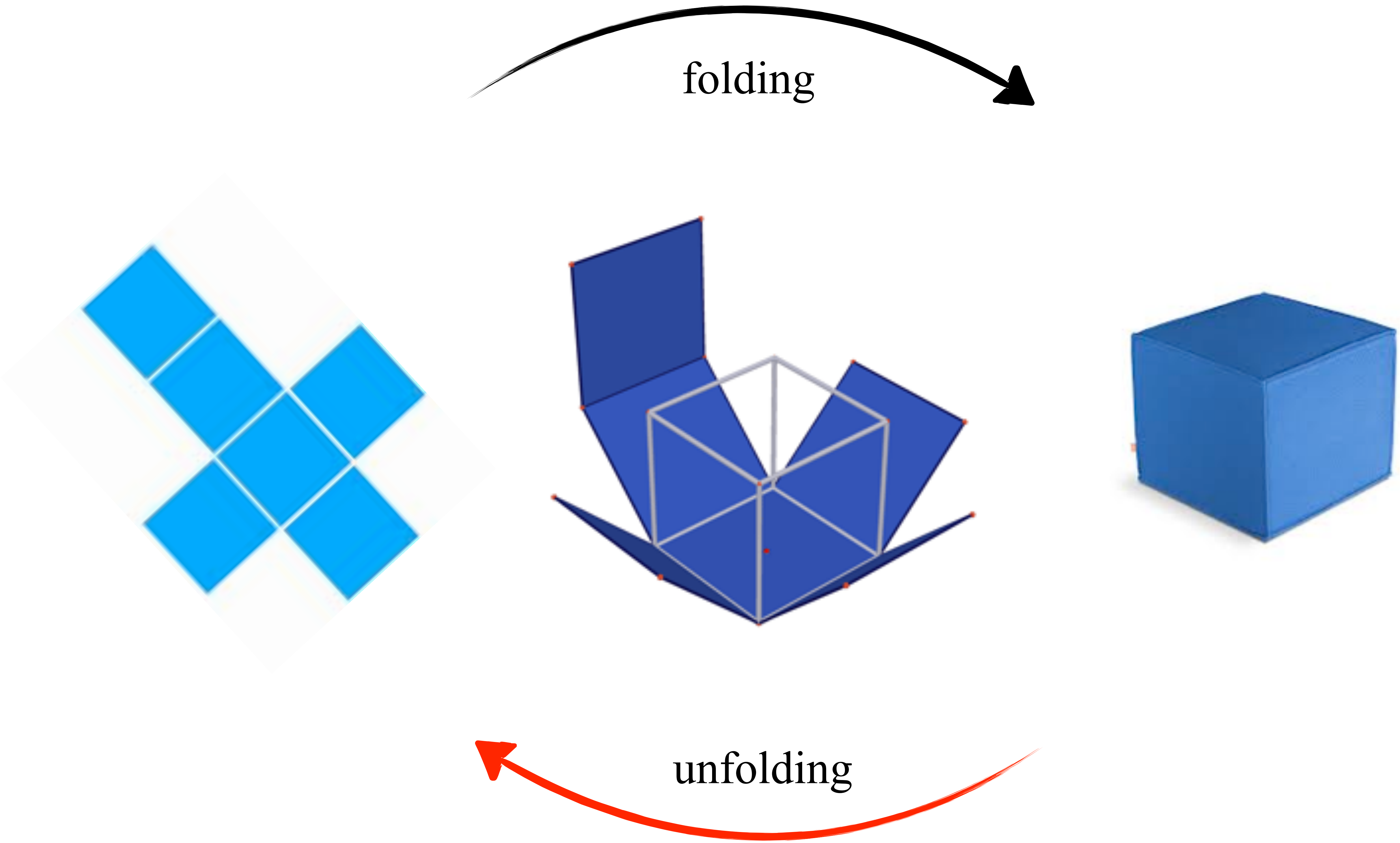



O'Rourke

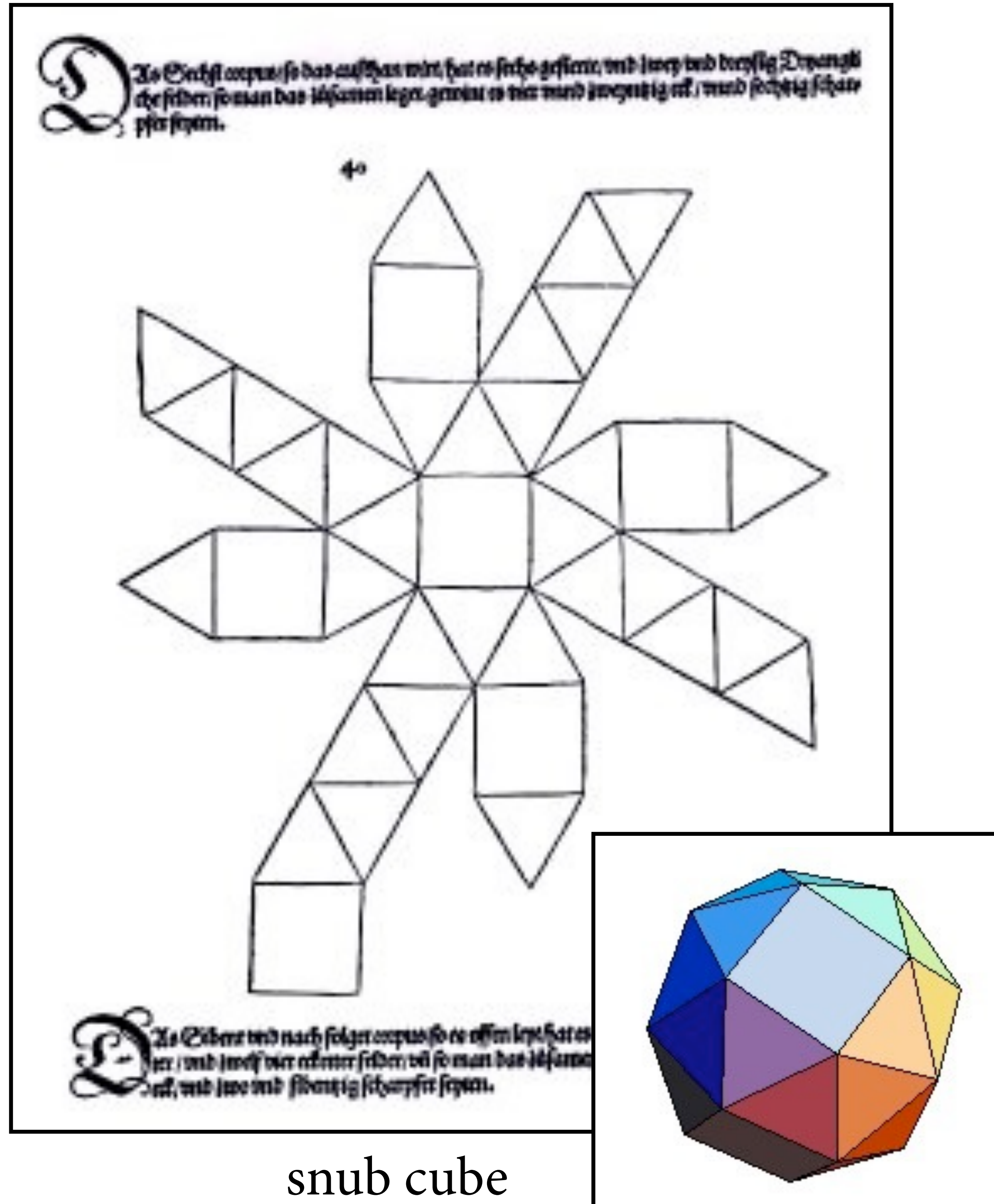
Shortest paths from point x to all points on the surface.

# An aside:  Folding and Unfolding



# Unfolding Polyhedra—Durer 1400's

Durer, 1498

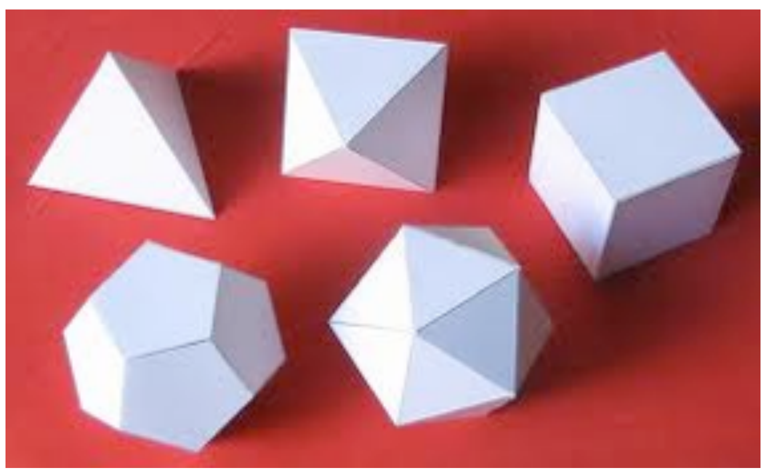


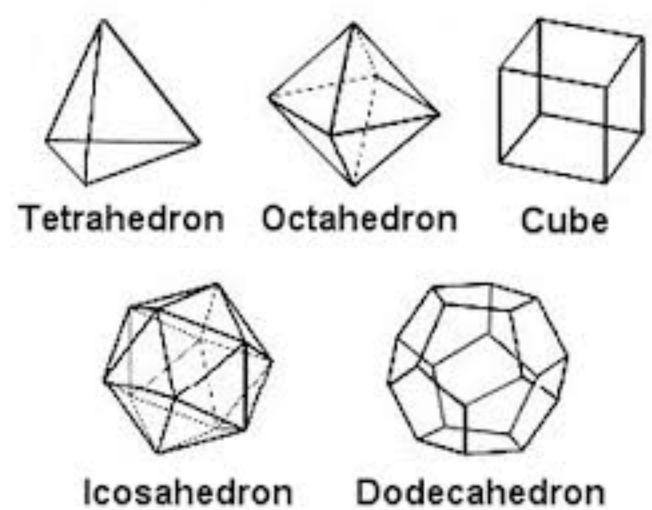
snub cube

# Open Question

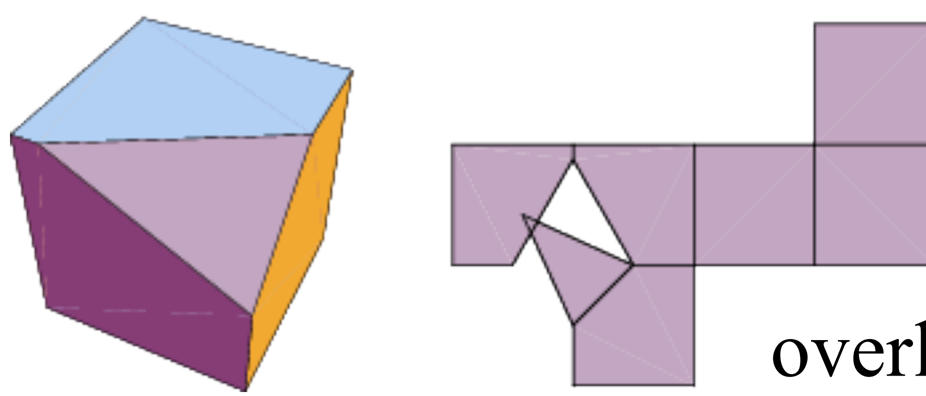
Can every convex polyhedron be unfolded into one piece without overlap by cutting edges?

good examples






bad example

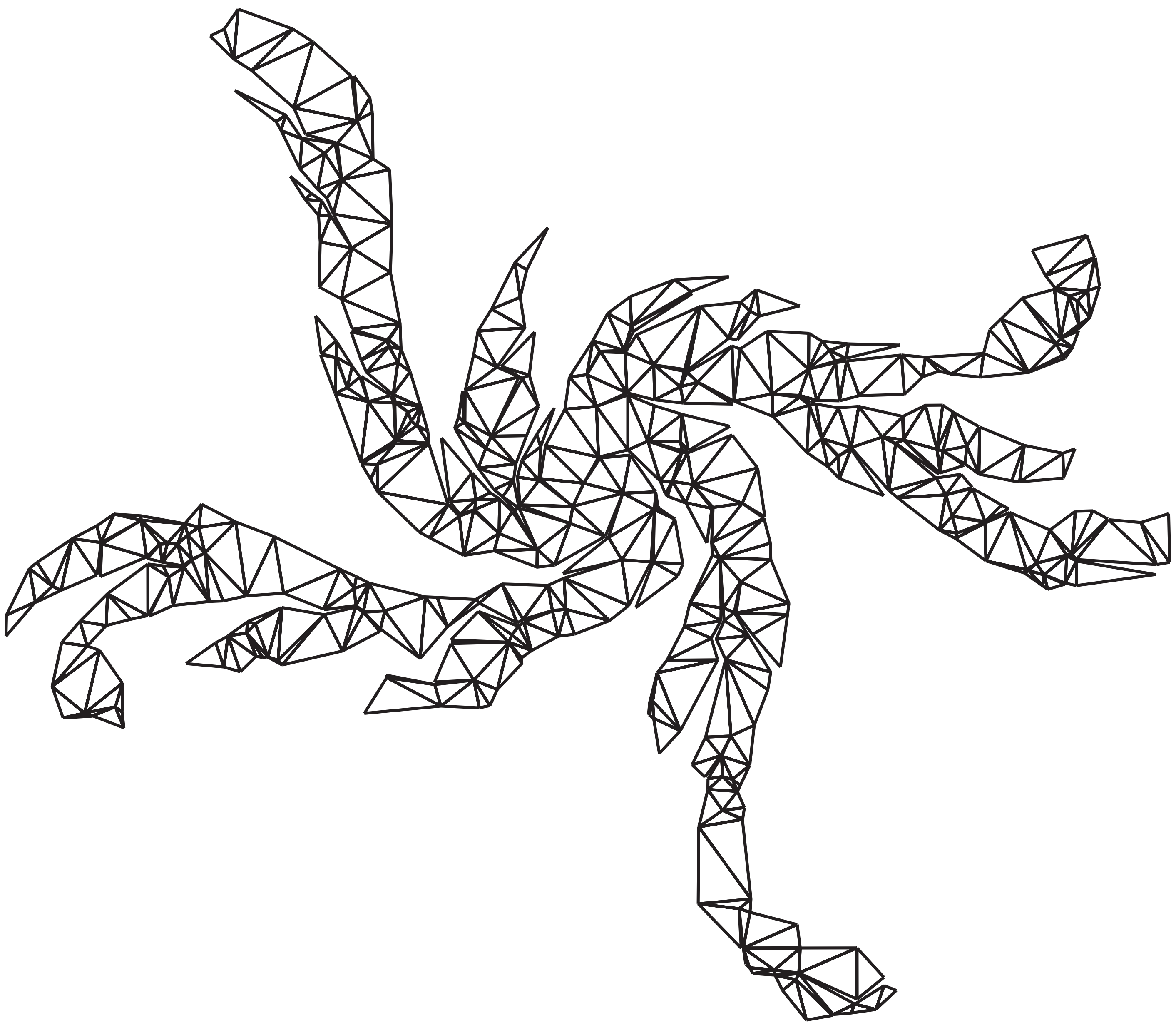


overlap

but there is a better way to cut this polyhedron

# Open Question

Can every convex polyhedron be unfolded into one piece without overlap by cutting edges?



every example tried succeeds

# Shortest Paths so far

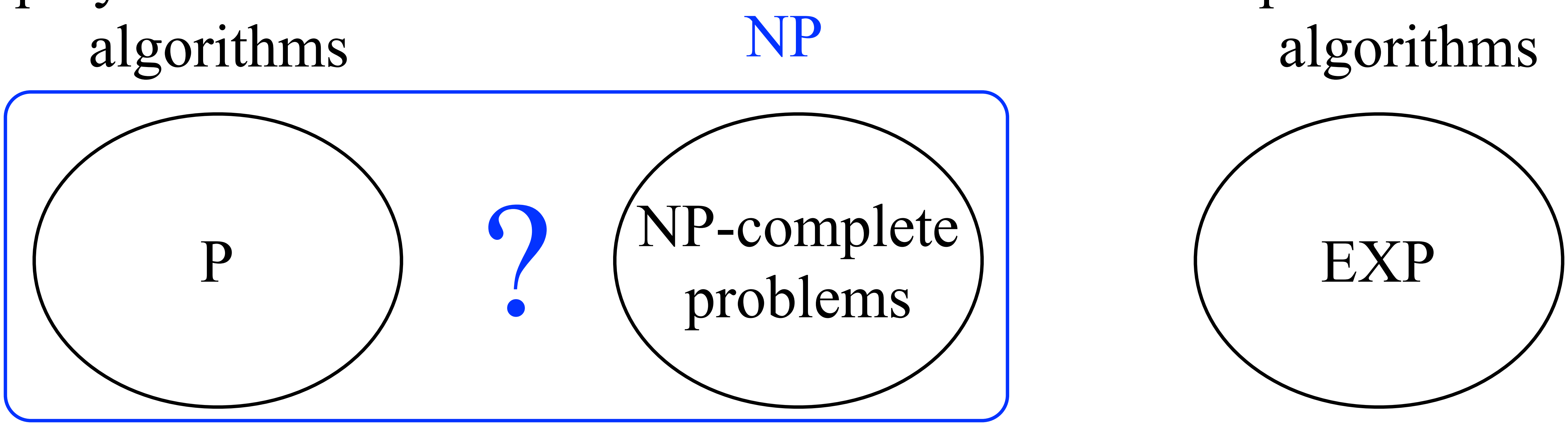
polynomial time algorithms for shortest paths in

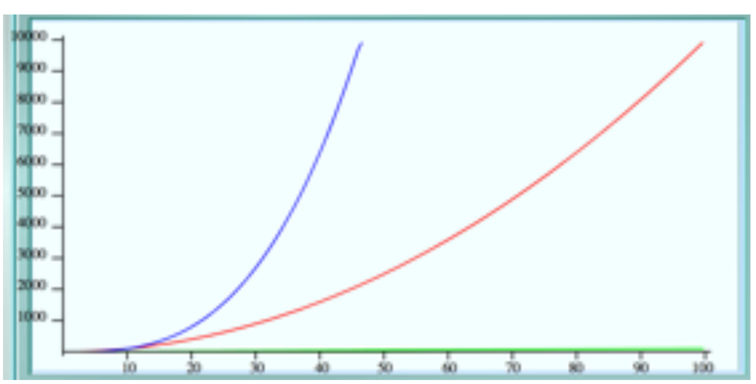
- graph
- polygon
- plane with polygonal obstacles
- polyhedron

# Hard and Easy Problems

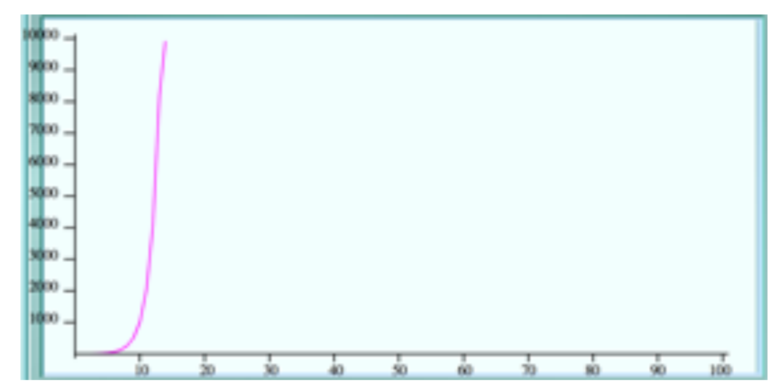
problems with polynomial-time algorithms

problems that require exponential-time algorithms



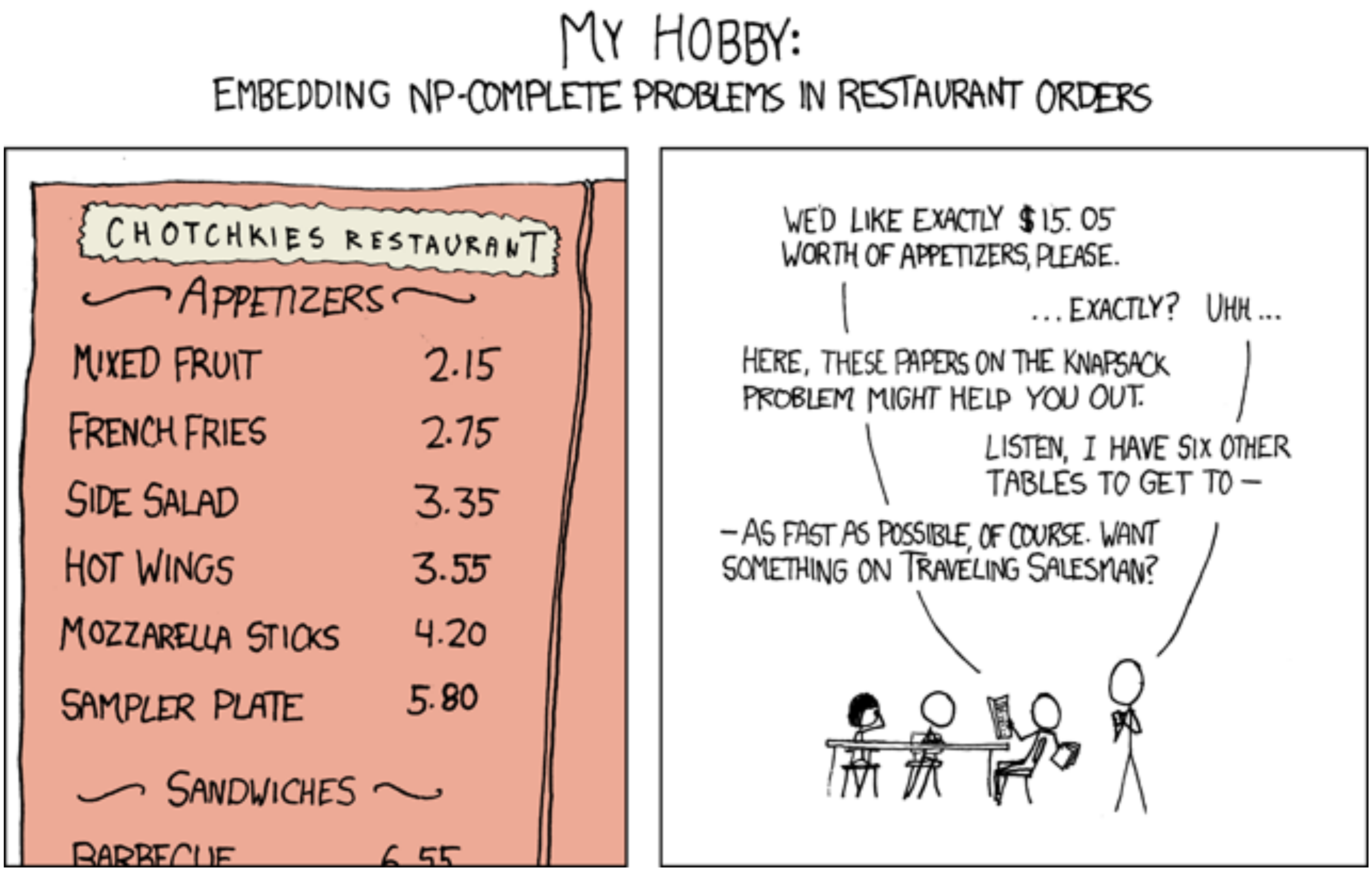



?



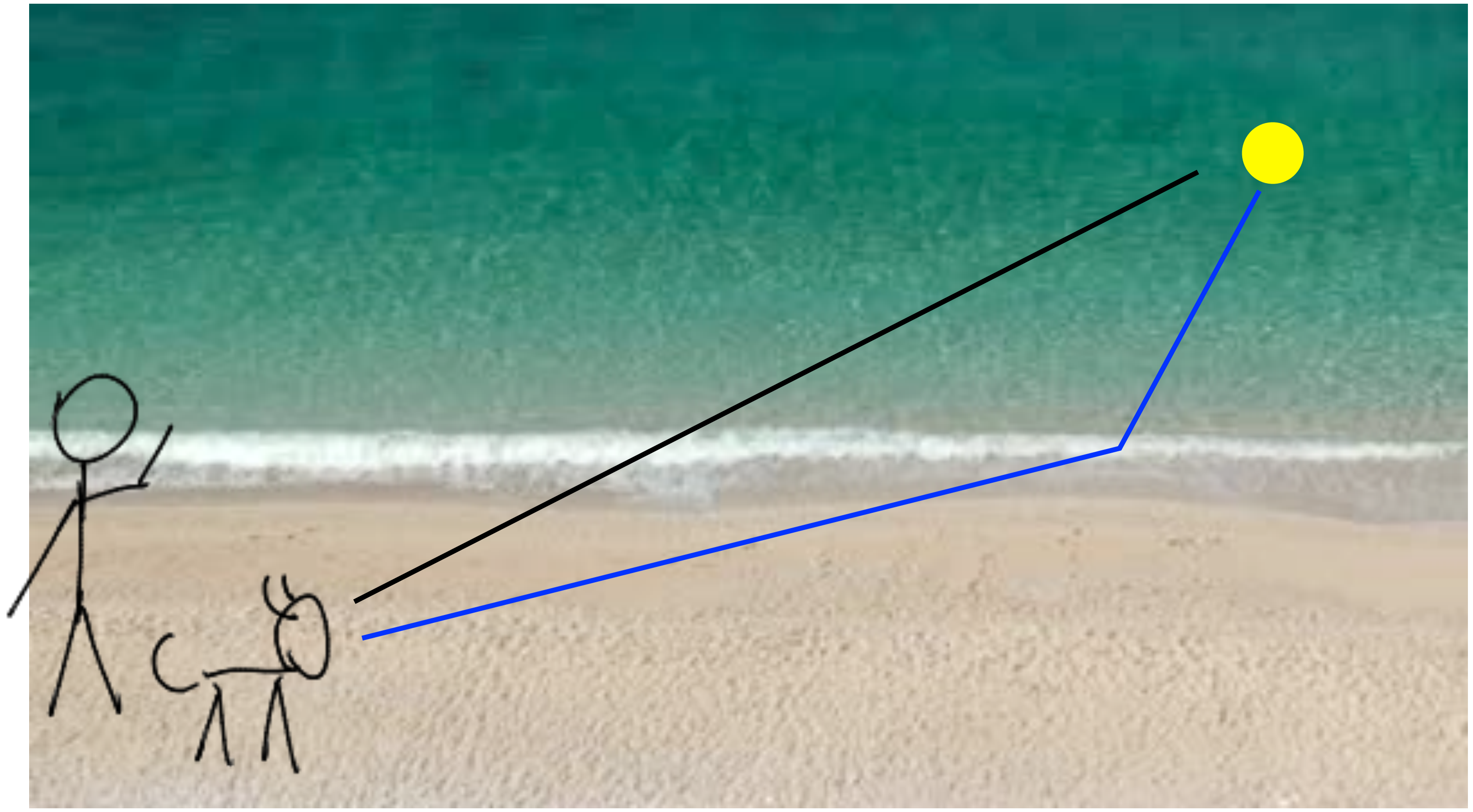
Clay Math Institute offers $1M prize to solve $P \stackrel{?}{=} NP$

# NP-complete

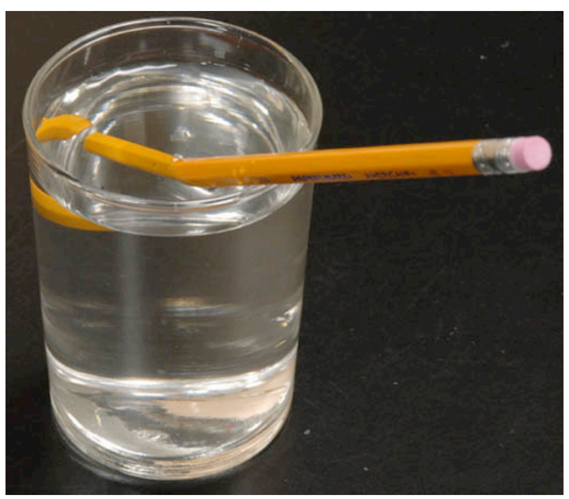


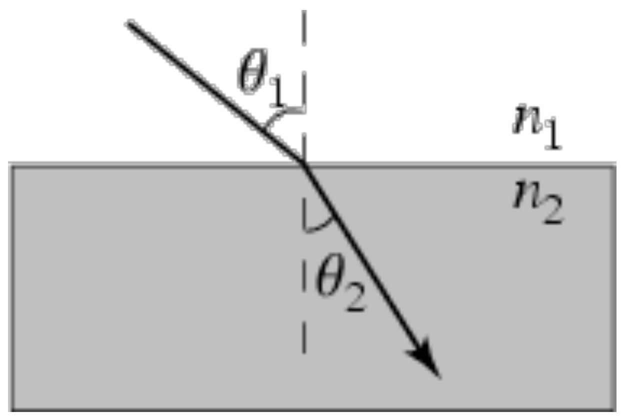
MY HOBBY:
EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS
CHOTCHKIES RESTAURANT
APPETIZERS
MIXED FRUIT 2.15
FRENCH FRIES 2.75
SIDE SALAD 3.35
HOT WINGS 3.55
MOZZARELLA STICKS 4.20
SAMPLER PLATE 5.80
SANDWICHES
WE'D LIKE EXACTLY $15.05 WORTH OF APPETIZERS, PLEASE.
...EXACTLY? UHH...
HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT.
LISTEN, I HAVE SIX OTHER TABLES TO GET TO –
– AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?

# Harder Shortest Path Problems

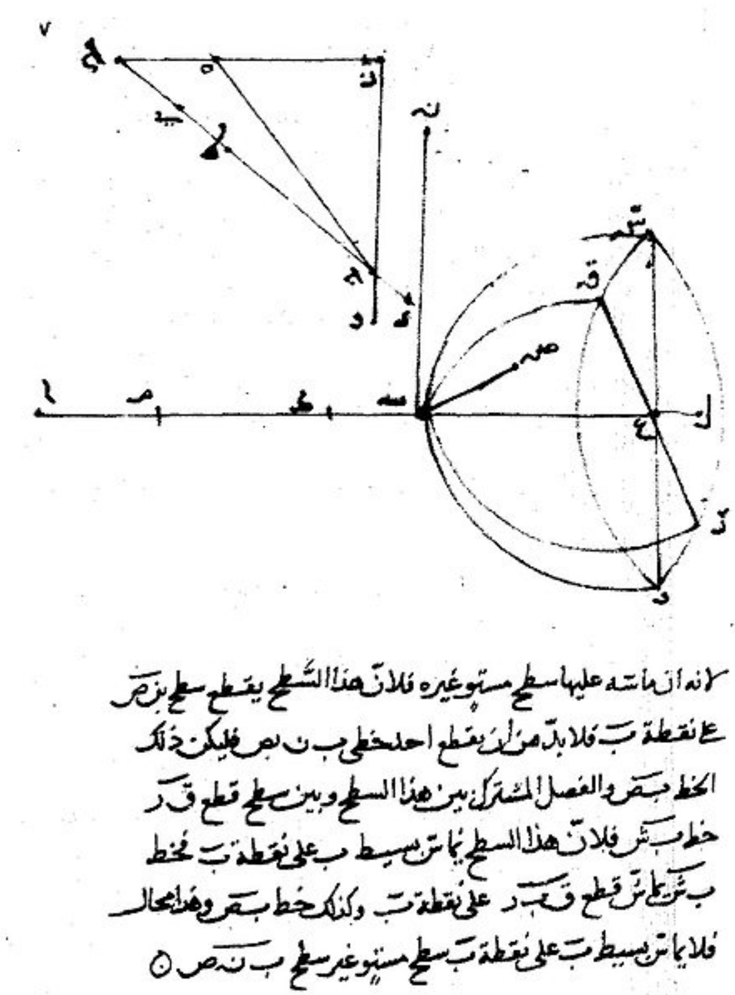


Snell's law



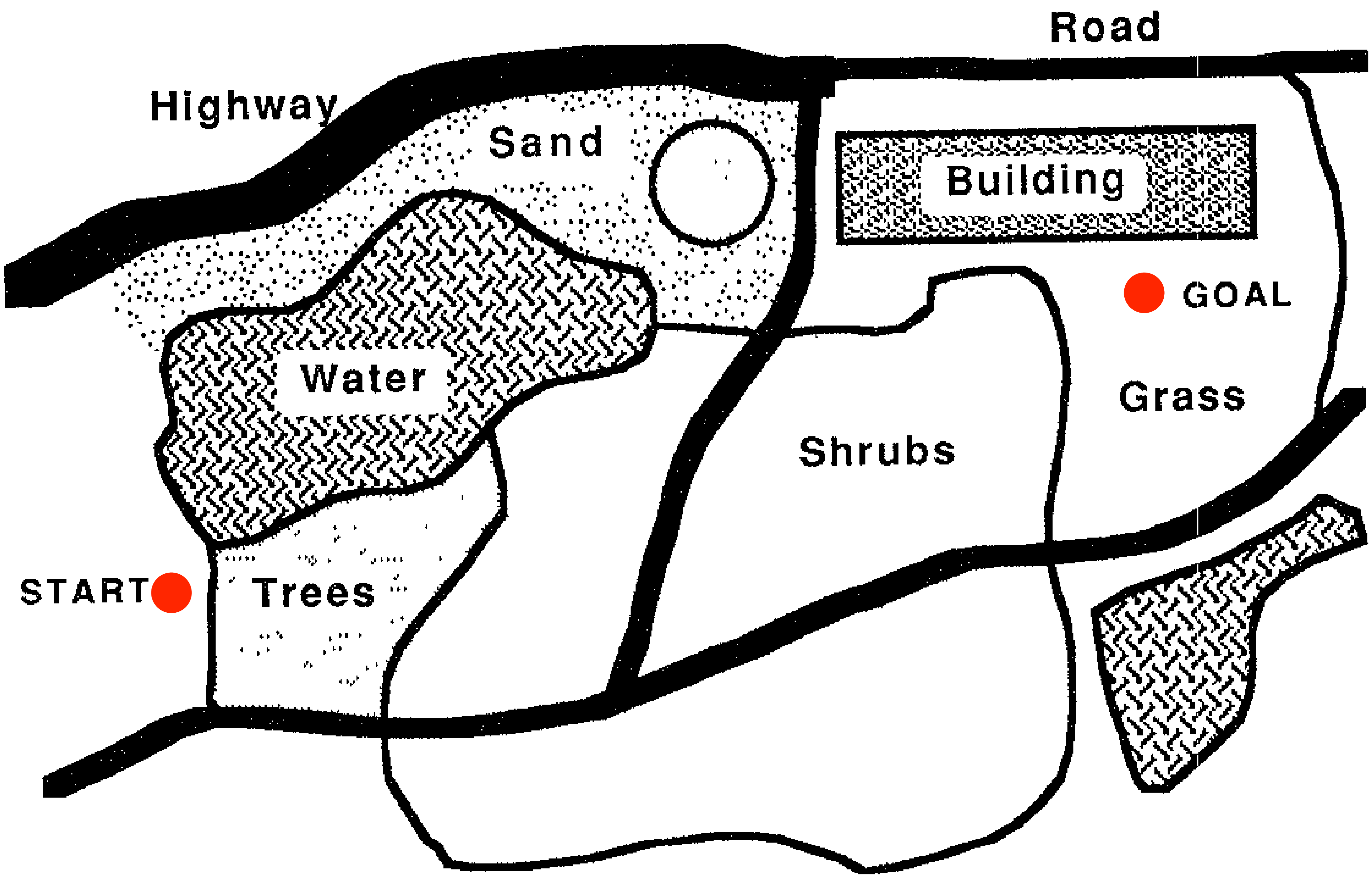



$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$



- Ibn Sahl, (Baghdad), On Burning Mirrors and Lenses, 984
- Willebrord Snellius, 1621
- René Descartes, 1637

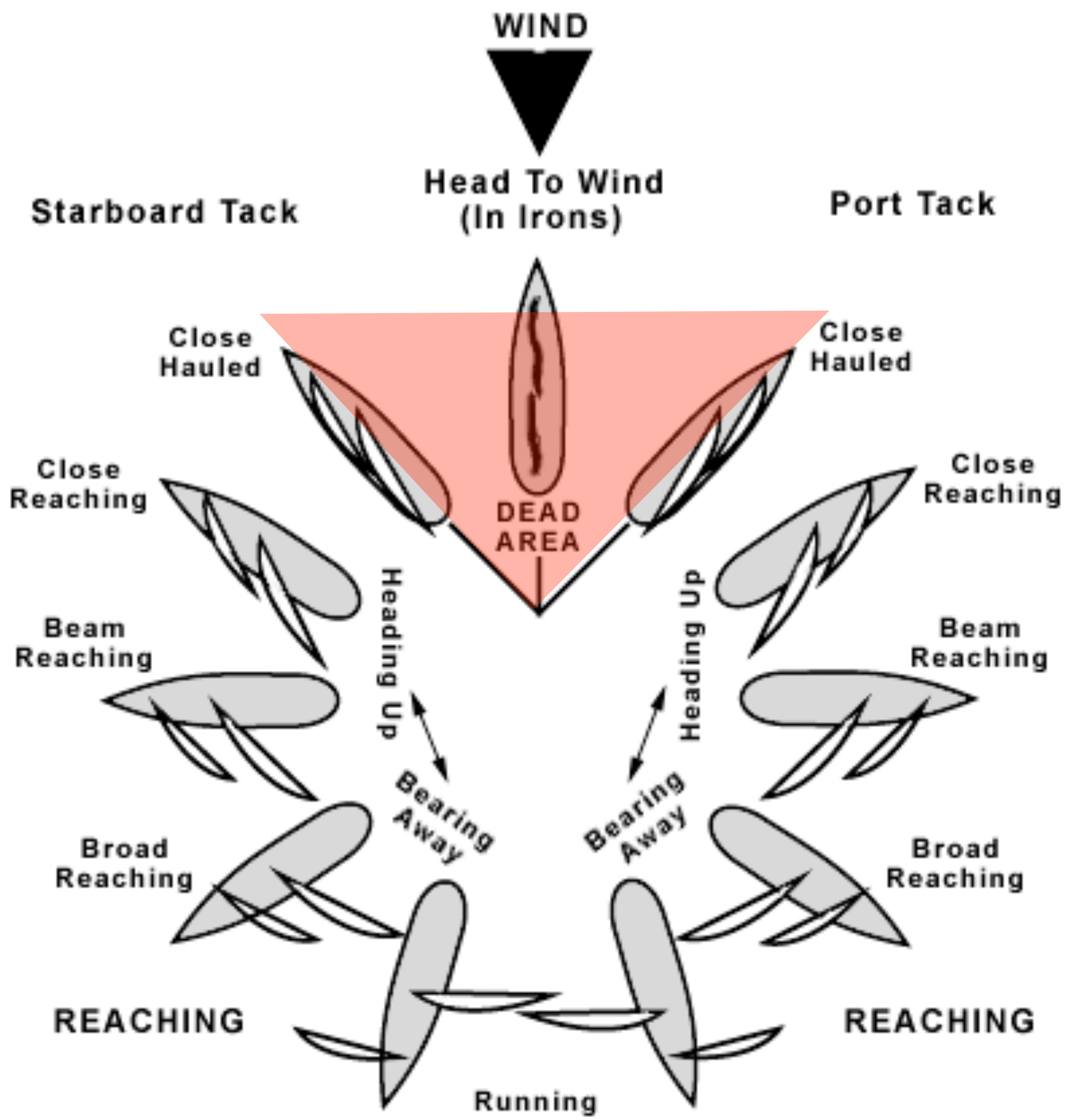
# Harder Shortest Path Problems

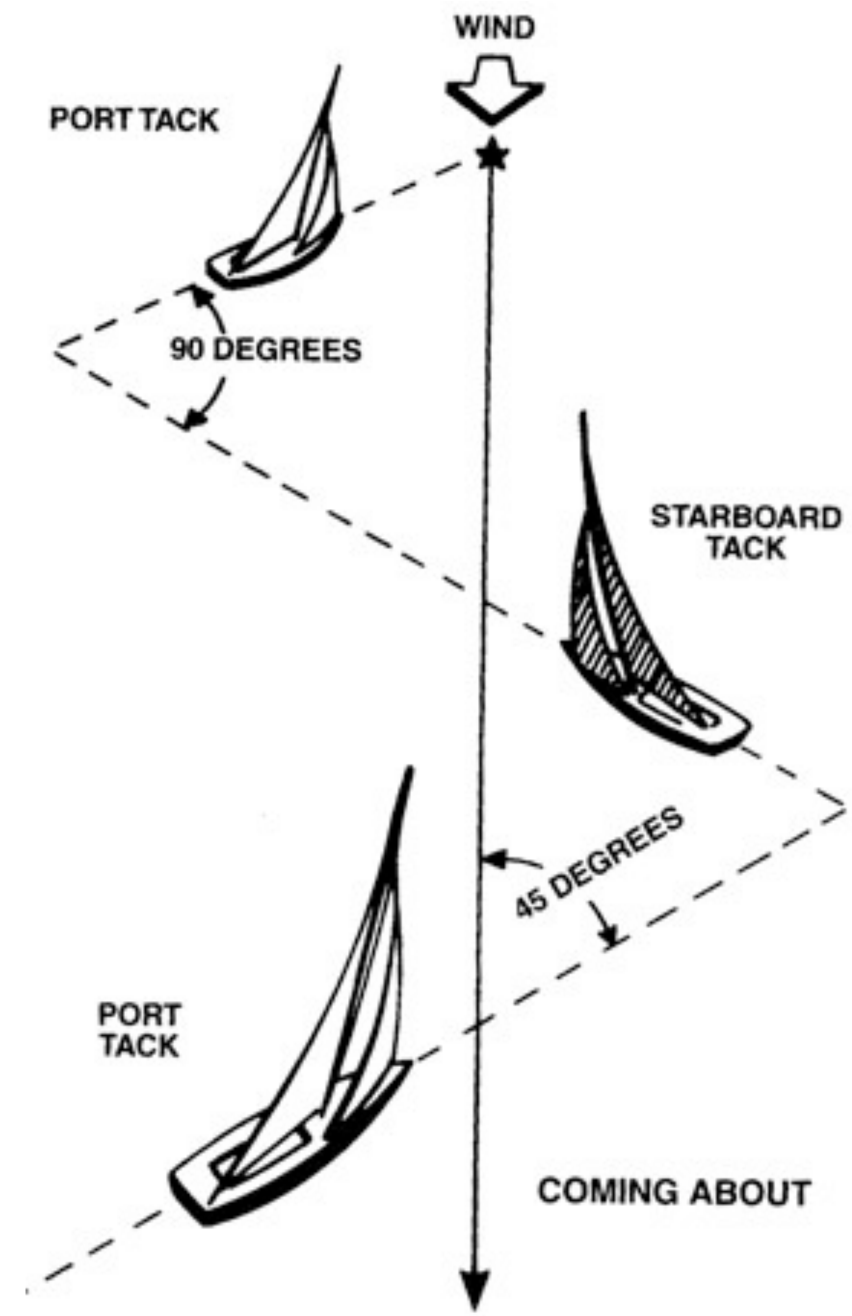



The Weighted Region Problem -- Mitchell and Papadimitriou '91

OPEN: is this problem in P? NP-complete?

# Shortest Anisotropic Paths

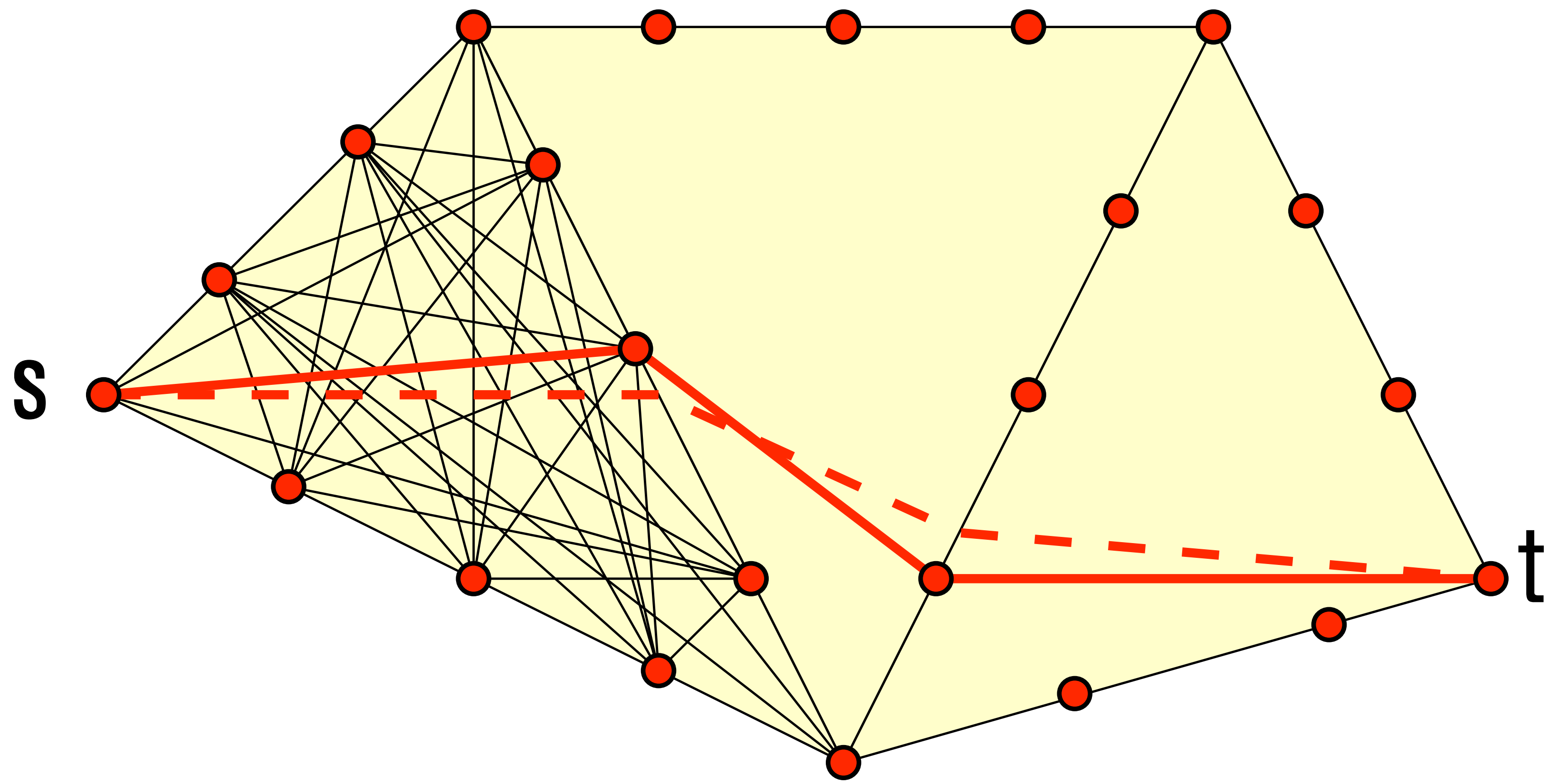



OPEN: is this problem in P? NP-complete?

# Approximation Algorithm (Steiner Point Approach)

for weighted region and some special cases of anisotropic




add many "Steiner" points and model as shortest path in a graph

# Shortest Descending Paths on Terrains

joint work with Mustaq Ahmed

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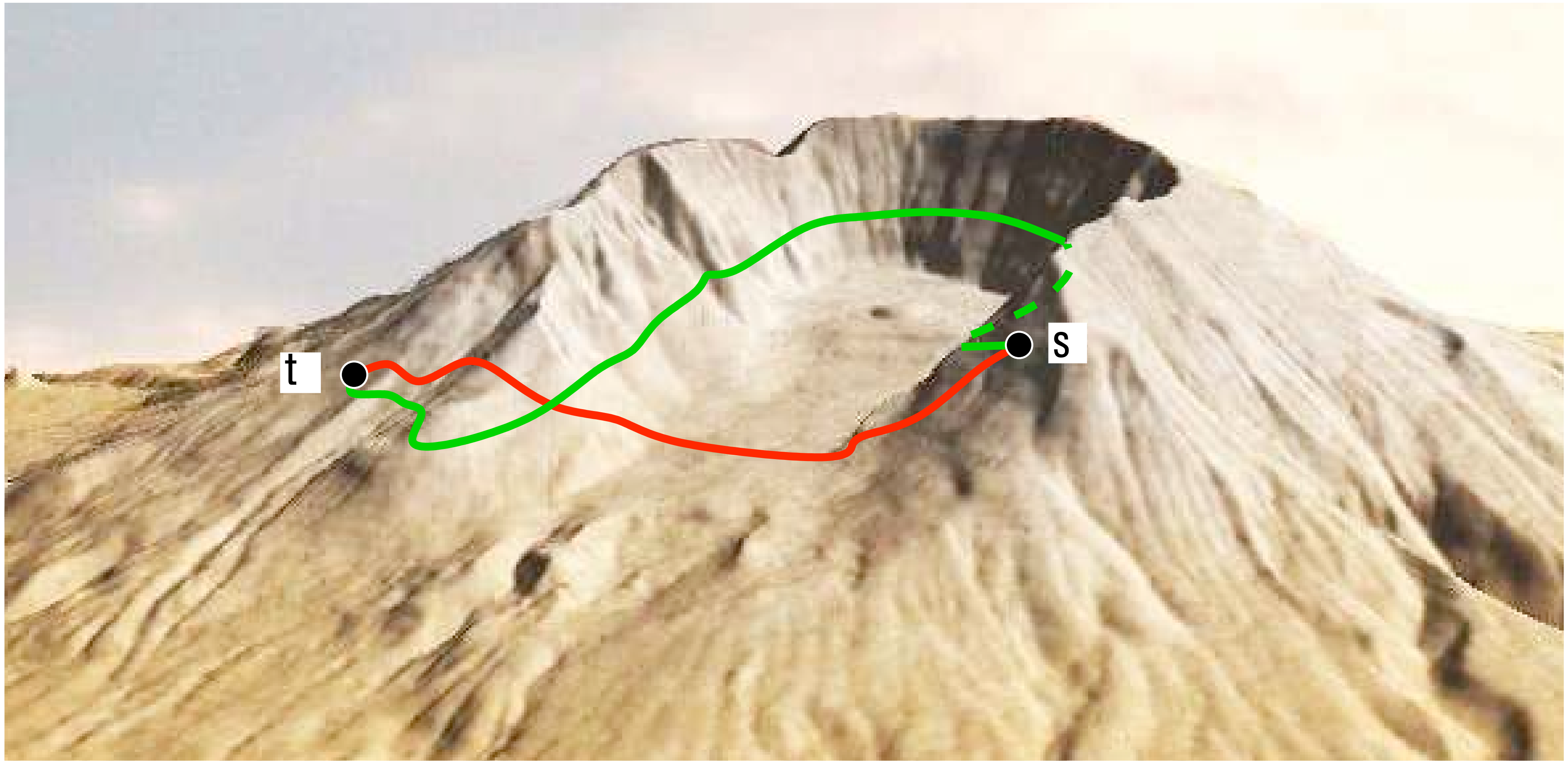
joint work with Mustaq Ahmed



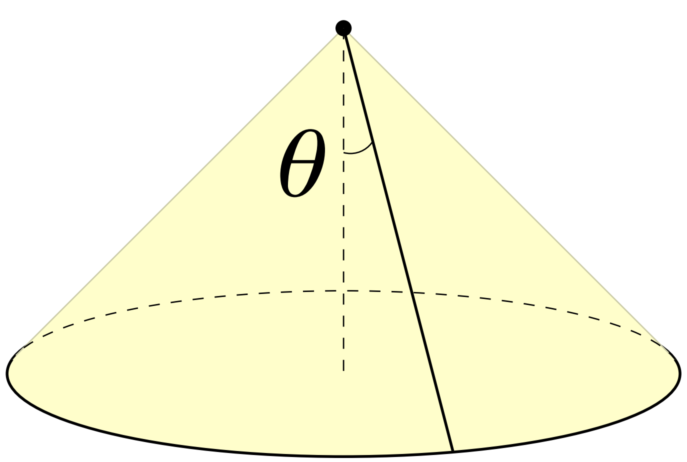

OPEN: is this problem in P? NP-complete?

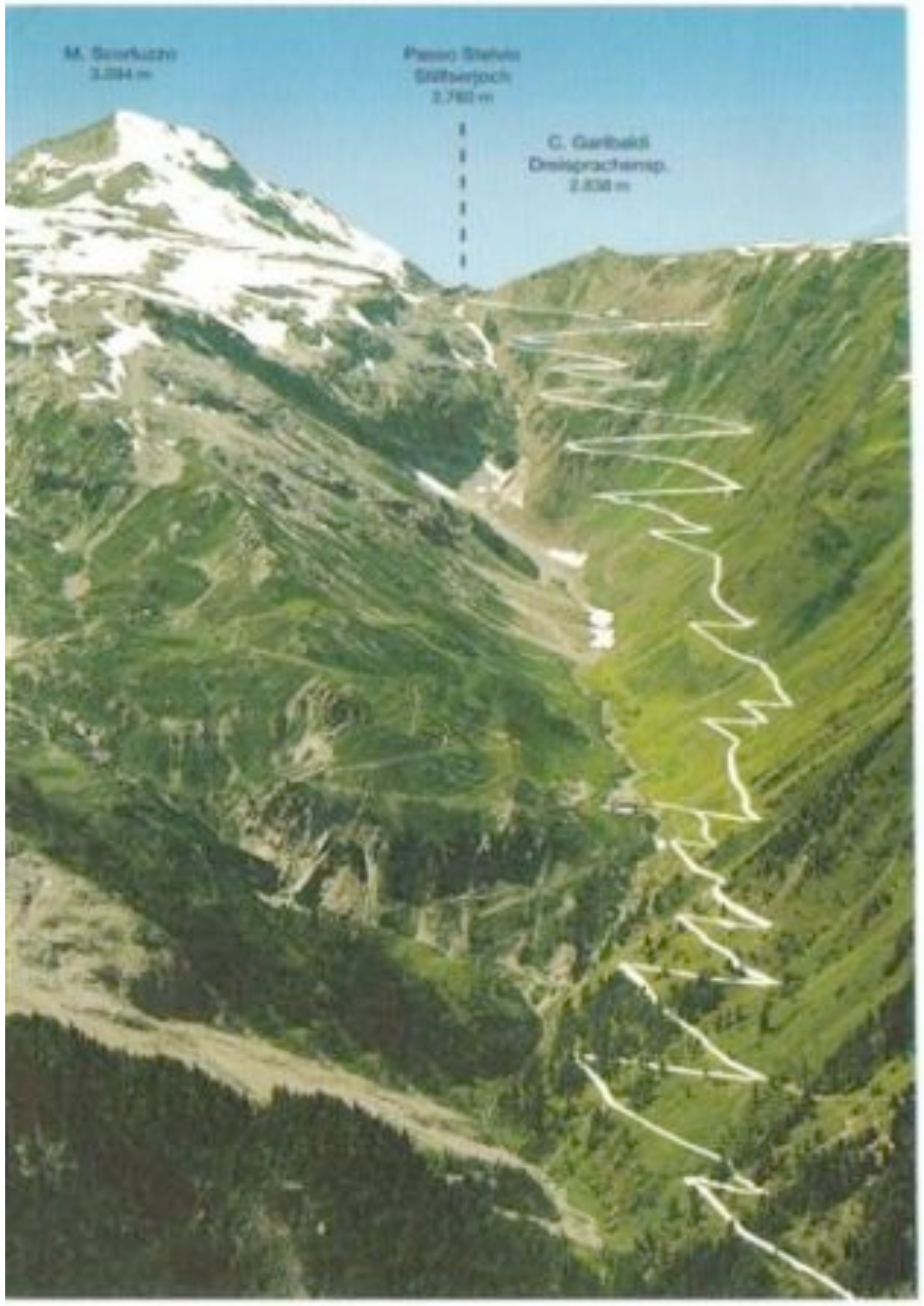
# Shortest Gently Descending Paths on Terrains

joint work with Mustaq Ahmed

Find a shortest path that descends, but not too *steeply*.

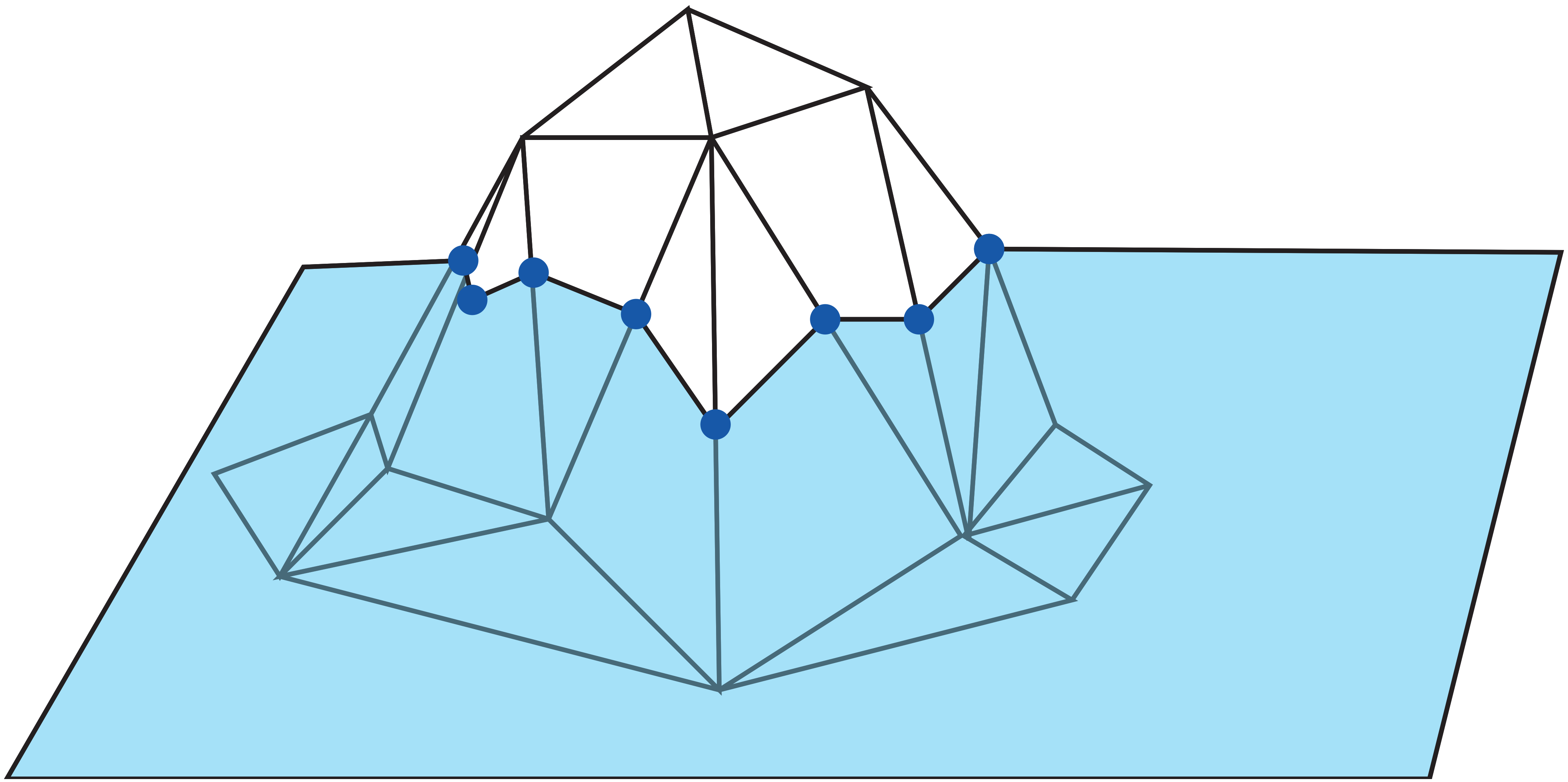
*steep* = lies in this cone



OPEN: is this problem in P? NP-complete?

# Approximation Algorithm (Steiner Point Approach)



add many "Steiner" points and model as shortest path in a graph

# Hard and Easy Problems

problems with polynomial-time algorithms

NP

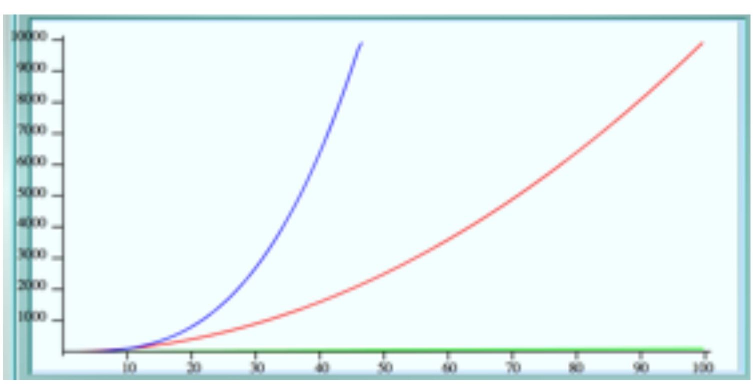
problems that require exponential-time algorithms

P

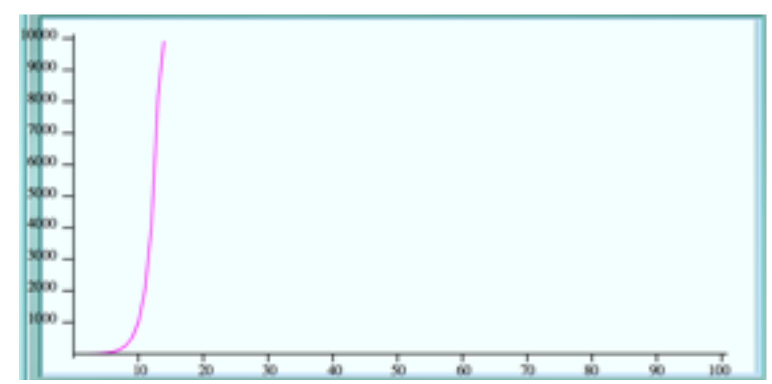
?

NP-complete problems

EXP



?



Clay Math Institute offers \$1M prize to solve $P \stackrel{?}{=} NP$

# Shortest Paths in 3D Space

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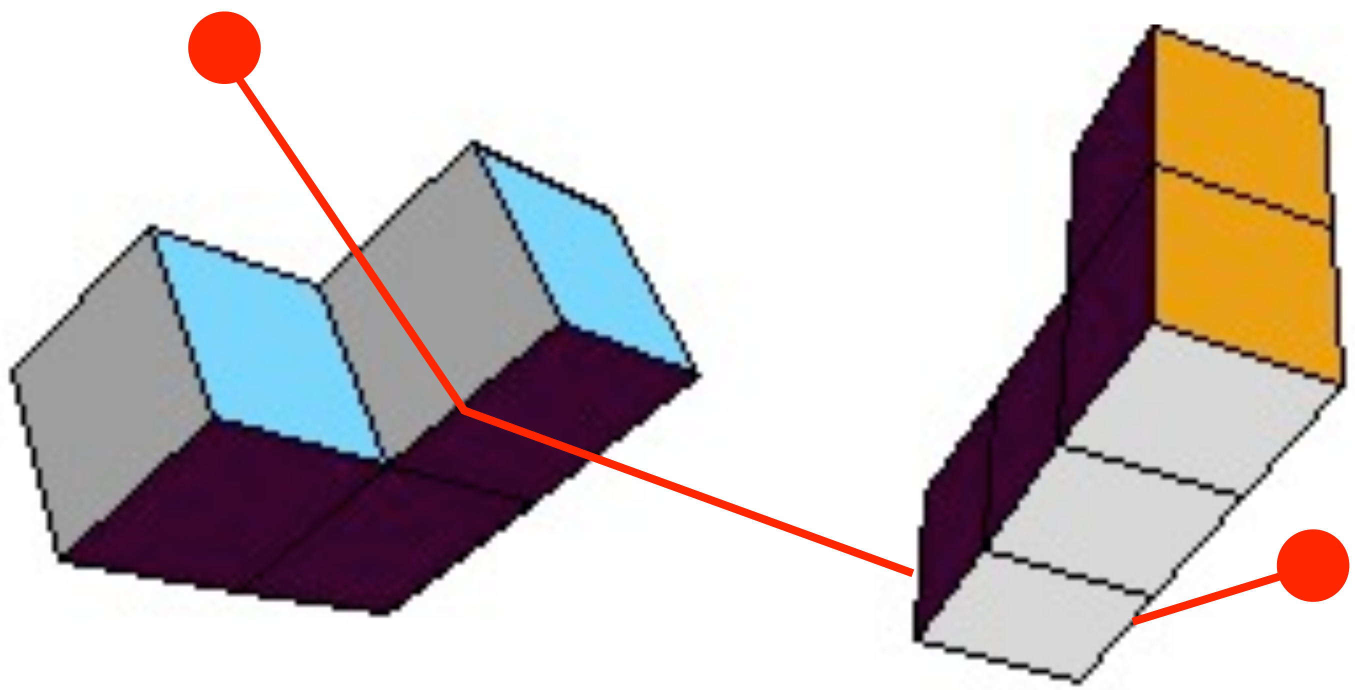
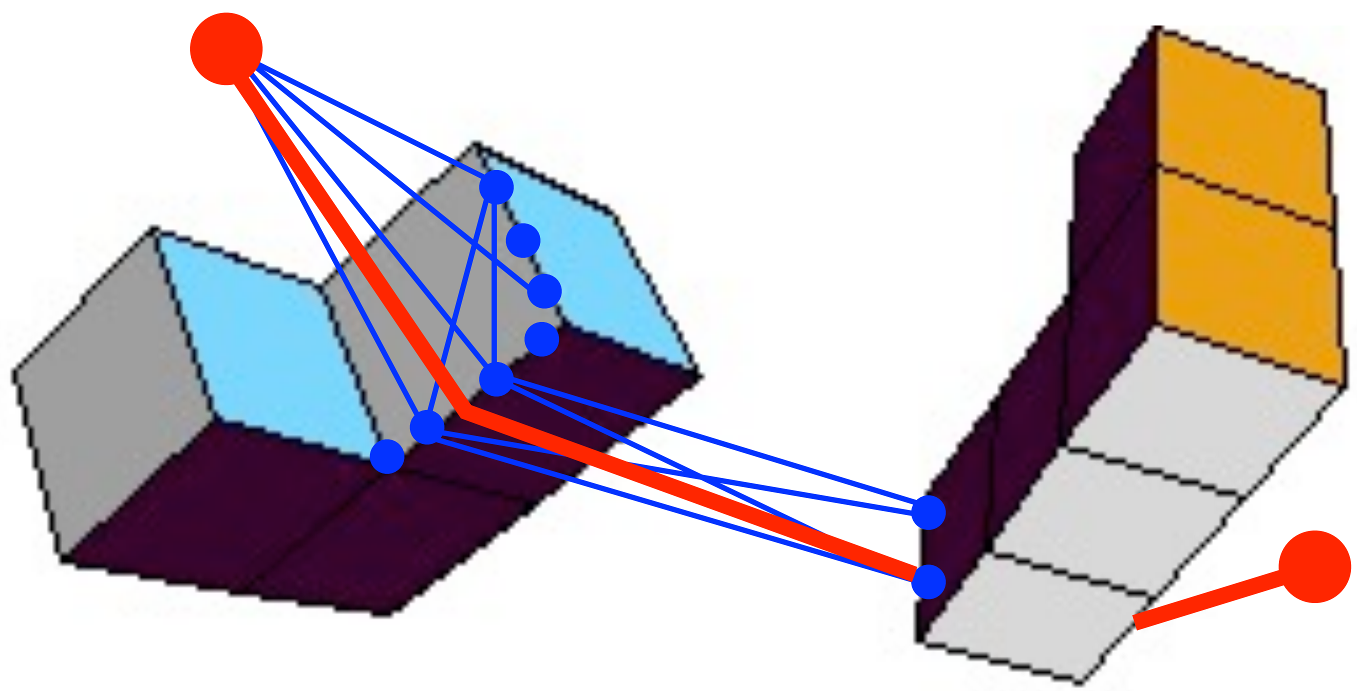


This problem is NP-hard.

Canny & Reif, 1987

# Approximation Algorithm (Steiner Point Approach)

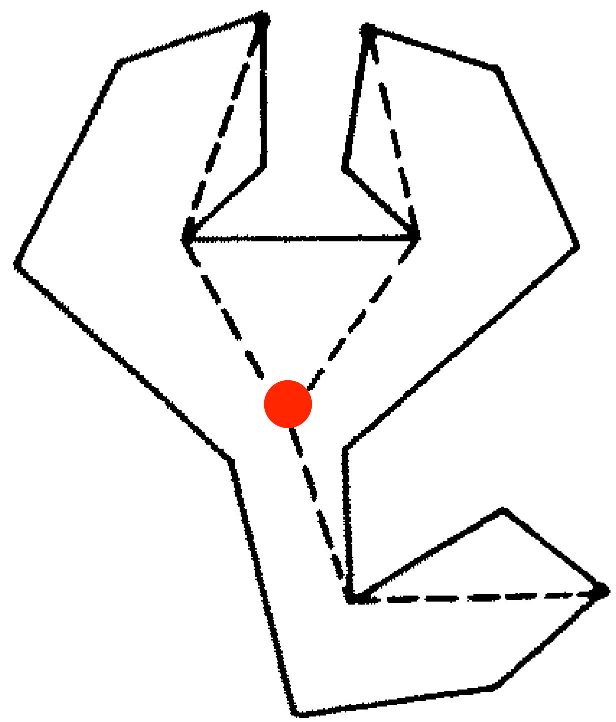
Papadimitriou ’85



Add many Steiner points and model as shortest path in a graph.

# New Results on Shortest Paths

center of a polygon



find a point to minimize the maximum distance to any point

$O(n)$ time algorithm, June 2015

# Research Topics

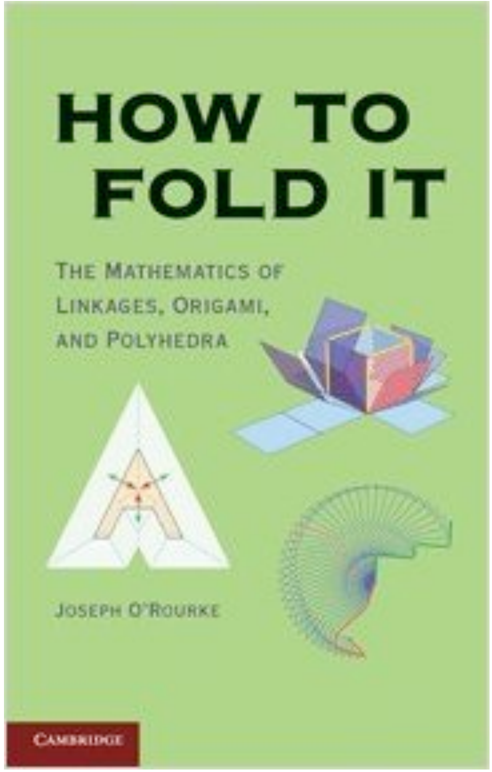
- practical shortest path methods for large graphs/maps
- shortest paths as graphs change (“dynamic” graphs)
- center and diameter

# More Information

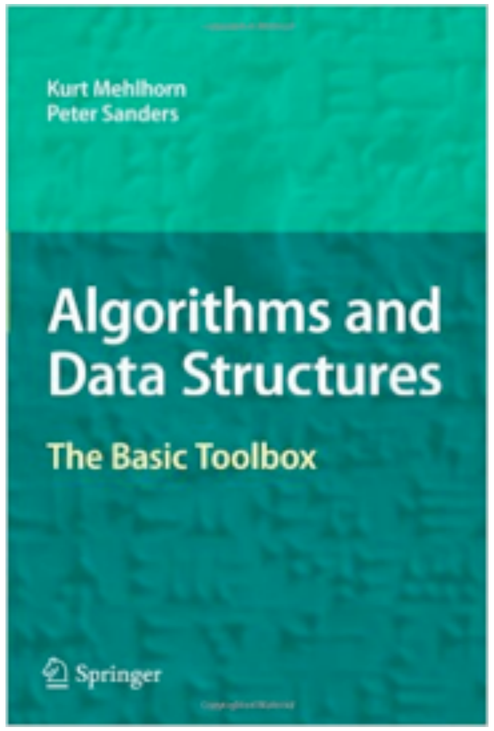
shortest paths course https://cs.uwaterloo.ca/~alubiw/CS860.html

Geometric Shortest Paths and Network Optimization,
survey by Joseph Mitchell

book on folding

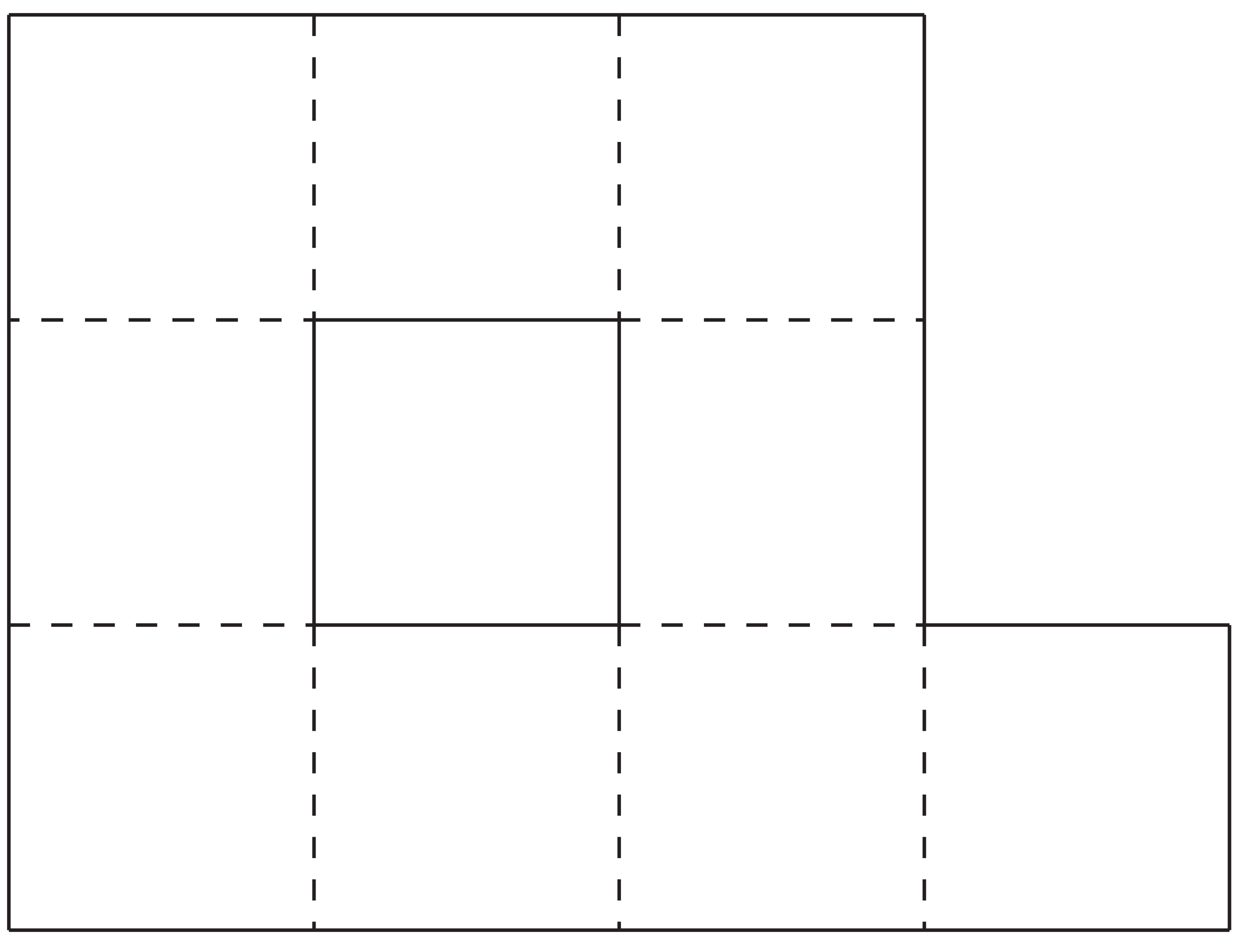



chapter on shortest paths:



I offer a free copy of “How to Fold It” to the first person who solves this folding puzzle (which has nothing to do with shortest paths)

Cut out this shape (with 9 unit squares) and fold it into a cube only folding on the dashed lines.



THE END